SISSA - ISAS
INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Area of Mathematics

Interaction functionals, Glimm approximations and Lagrangian structure of BV solutions for Hyperbolic Systems of Conservation Laws

Ph.D. Thesis

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ACADEMIC YEAR 2014/2015

Il presente lavoro costituisce la tesi presentata da Stefano Modena, sotto la direzione del Prof. Stefano Bianchini, al fine di ottenere l'attestato di ricerca post-universitaria *Doctor Philosophiæ* presso la SISSA, Curriculum in Matematica Applicata, Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della Sissa pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di *Dottore di Ricerca in Matematica*.

Trieste, Anno Accademico 2014-2015.

Ringraziamenti

Questa tesi rappresenta la conclusione del corso di dottorato triennale che ho svolto presso la Sissa di Trieste ed è una gradita occasione per ringraziare tutti coloro che con il loro aiuto e la loro vicinanza mi hanno permesso di raggiungere questo traguardo.

In primo luogo desidero ringraziare il mio relatore, il prof. Stefano Bianchini. Lavorare sotto la sua guida è stato per me un grande onore e un grande privilegio. Il continuo sostegno e la generosa disponibilità con cui mi ha sempre seguito in questi anni sono stati fondamentali per me, per la mia formazione e per la riuscita di questa tesi. Il suo esempio è stato un costante insegnamento attraverso il quale mi sono potuto avvicinare alla possibilità di cogliere il senso profondo di cosa significhi "fare matematica".

Durante il mio percorso universitario ho avuto la possibilità di conoscere molti altri professori e molti colleghi: da tutti ho imparato molto e a tutti va il mio ringraziamento. In modo particolare desidero ringraziare il prof. Alberto Bressan per avermi ospitato alla Penn State University e per il tempo che mi ha dedicato in occasione di alcuni interessantissimi colloqui; il prof. Fabio Ancona per la vicinanza e la disponibilità che mi ha costantemente dimostrato e per le svariate e stimolanti discussioni avute sugli argomenti di questi tesi; il prof. Daniele Del Santo per la benevolenza con cui mi ha sempre accompagnato e i preziosi suggerimenti che sempre ha voluto donarmi.

Rivolgo un grazie cordiale a tutti i miei compagni di dottorato, in particolare agli amici del mio anno e ad Elio, Nikolay e Paolo con i quali ho condiviso splendidi momenti.

Infine, un ringraziamento del tutto speciale va alla mia famiglia: in primo luogo a mia moglie Ada, che mi è stata sempre vicina con amore, dolcezza e delicatezza, condividendo con me le gioie e le fatiche; e poi ai miei genitori, senza il cui sostegno e incoraggiamento, di certo, non sarei potuto arrivare sin qui.

A tutti va la mia più profonda gratitudine.

Trieste, 25 settembre 2015

Contents

Abstract

This thesis is a contribution to the mathematical theory of Hyperbolic Conservation Laws. Three are the main results which we collect in this work. The first and the second result (denoted in the thesis by Theorem A and Theorem B respectively) deal with the following problem. The most comprehensive result about existence, uniqueness and stability of the solution to the Cauchy problem

$$\begin{cases} u_t + F(u)_x = 0, \\ u(0,x) = \bar{u}(x), \end{cases} \tag{$\mathcal{C}$}$$

where $F : \mathbb{R}^N \to \mathbb{R}^N$ is strictly hyperbolic, $u = u(t,x) \in \mathbb{R}^N$, $t \geq 0$, $x \in \mathbb{R}$, $\text{Tot.Var.}(\bar{u}) \ll 1$, can be found in [BB05], where the well-posedness of ($\mathcal{C}$) is proved by means of vanishing viscosity approximations. After the paper [BB05], however, it seemed worthwhile to develop a *purely hyperbolic* theory (based, as in the genuinely nonlinear case, on Glimm or wavefront tracking approximations, and not on vanishing viscosity *parabolic* approximations) to prove existence, uniqueness and stability results. The reason of this interest can be mainly found in the fact that *hyperbolic* approximate solutions are much easier to study and to visualize than *parabolic* ones. Theorems A and B in this thesis are a contribution to this line of research. In particular, Theorem A proves an estimate on the change of the speed of the wavefronts present in a Glimm approximate solution when two of them interact; Theorem B proves the convergence of the Glimm approximate solutions to the weak admissible solution of ($\mathcal{C}$) and provides also an estimate on the rate of convergence. Both theorems are proved in the most general setting when no assumption on F is made except the strict hyperbolicity.

The third result of the thesis, denoted by Theorem C, deals with the Lagrangian structure of the solution to ($\mathcal{C}$). The notion of Lagrangian flow is a well-established concept in the theory of the transport equation and in the study of some particular system of conservation laws, like the Euler equation. However, as far as we know, the general system of conservation laws ($\mathcal{C}$) has never been studied from a Lagrangian point of view. This is exactly the subject of Theorem C, where a Lagrangian representation for the solution to the system ($\mathcal{C}$) is explicitly constructed. The main reasons which led us to look for a Lagrangian representation of the solution of ($\mathcal{C}$) are two: on one side, this Lagrangian representation provides the continuous counterpart in the exact solution of ($\mathcal{C}$) to the well established theory of wavefront approximations; on the other side, it can lead to a deeper understanding of the behavior of the solutions in the general setting, when the characteristic field are not genuinely nonlinear or linearly degenerate.

The publications related to the results of this thesis are [BM14a], [BM14b], [BM15b], [Mod15a], [Mod15b] for Theorem A, [MB15] for Theorem B and [BM15a] for Theorem C.

Introduction

This thesis is a contribution to the mathematical theory of Hyperbolic Conservation Laws. Aim of this Introduction is to present an overview of the three main results of this thesis, discuss how they are related to the general theory of conservation laws and give an outline of the techniques used to prove them.

A *system of conservation laws* in one space dimension is a system of PDEs of the form

$$u_t + F(u)_x = 0, \tag{1}$$

where $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}^N$ is the unknown and $F : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}^N$ is a given smooth (say C^3) map, called *flux*, defined on a neighborhood Ω of a compact set $K \subseteq \mathbb{R}^N$.
Equation (1) is usually coupled with an initial datum

$$u(0, x) = \bar{u}(x), \tag{2}$$

where $\bar{u} : \mathbb{R} \to \mathbb{R}^N$ is a given function. It is customary to assume that the system (1) satisfies the *strict hyperbolicity condition*, i.e. the Jacobian $DF(u)$ of F has N distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_N(u) \tag{3}$$

in each point $u \in \Omega$ of its domain. We will also assume, for simplicity, that the initial datum $\bar{u}$ is identically zero out of a compact set. This is just a technical assumption which can always be removed.

Systems of conservation laws are very important for applications. For instance, they are widely used to express the fundamental balance laws of continuum physics (see [Daf05]), when small viscosity or dissipation effects are neglected. As an example, the Euler equation for a compressible, non-viscous gas takes the form of a system of three conservation laws, where the unknowns are the mass specific volume of the gas and its velocity. Conservation laws are also used in several other fields, like biology, elastodynamics, rigid heat conductors, superfluids or traffic flow models. In the latter case, for instance, the unknown $u(t, x) \in \mathbb{R}$ is the density of car at time t on the point x and the map $F = F(u)$ represents the flux of cars as a function of their density u. In this thesis, however, we will not focus on a single example of conservation laws; on the contrary, we will develop results which hold for the general system (1), (2) without any assumption on the flux F except the strict hyperbolicity, and which are based on the careful analysis of the wave interactions.

In order to better understand how the three main results of this thesis fit in the general theory of Hyperbolic Conservation Laws, we consider worthwhile to present first an extended, even if far from complete, history of general existence, uniqueness and stability theory for the Cauchy problem (1), (2). Then we will give a general, not technical, overview of the three main results of the thesis, providing also the precise statement of the three main theorems we will prove in this thesis. In the meanwhile, also further research directions will be proposed.

History of the previous results

Admissibility criteria. It is well known that, due to the nonlinear dependence of the characteristic speeds $\lambda_k(u)$ on the state variable u, waves may compress and classical (smooth) solution to (1), (2) can develop discontinuities in finite time. On the other side, the notion of distributional solution is too weak to guarantee the uniqueness. For this reasons the notion of solution which is typically used is the following one.

DEFINITION 1. A map $u : [0,\infty)\times\mathbb{R} \to \mathbb{R}^N$ belonging to L^1_{loc} is said to be *a weak solution* of the Cauchy problem (1), (2) if:

(1) u satisfies the equation (1) in the sense of distributions;
(2) u is continuous as a map $[0,\infty) \to L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$;
(3) at time $t = 0$, u satisfies the initial datum, i.e. $u(0,x) = \bar{u}(x)$;
(4) u satisfies some additional admissibility criteria, which come from physical or stability considerations and guarantee the uniqueness of the solution.

Many admissibility criteria have been proposed in the literature: just to name a few, the Lax-Liu condition on shocks (see [**Lax57**, **Liu74**, **Liu75**]), the entropy condition (see [**Lax71**]), the vanishing viscosity criterion (see [**BB05**]). We do not want to enter into details: the interested reader can refer to the cited literature and to the book [**Bre00**].

The Riemann problem. The basic ingredient to solve the Cauchy problem (1), (2) is the solution of the Riemann problem, i.e. the Cauchy problem when the initial datum has the simple form

$$u(0,x) = \bar{u}(x) = \begin{cases} u^L & \text{if } x < 0, \\ u^R & \text{if } x \geq 0. \end{cases} \tag{4}$$

The solution of the Riemann problem (1), (4) was obtained first by Peter Lax in 1957 [**Lax57**], under the assumption that each characteristic field is either *genuinely non linear* (GNL), i.e.

$$\nabla\lambda_k(u)\cdot r_k(u) \neq 0$$

for every u or *linearly degenerate* (LD), i.e.

$$\nabla\lambda_k(u)\cdot r_k(u) = 0$$

for every u. As usual, we are denoting by $r_1(u),\dots,r_n(u)$ the right eigenvalues (normalized to 1) associated to $\lambda_1(u),\dots,\lambda_n(u)$ respectively:

$$Df(u)r_k(u) = \lambda_k(u)r_k(u), \qquad \text{for every } k = 1,\dots,n \text{ and for every } u \in \Omega.$$

In this case, if $|u^R - u^L| \ll 1$, using Implicit Function Theorem, one can find intermediate states $u^L = \omega_0,\ \omega_1,\dots,\omega_n = u^R$ such that each pair of adjacent states (ω_{k-1},ω_k) can be connected by either a shock or a rarefaction wave of the k-th family (if GNL) or by a contact discontinuity of the k-th family (if LD). We will refer to shocks, rarefactions and contact discontinuities as *wavefronts*, rather than wave: we will indeed use the word "wave" to describe slightly different objects. The complete solution to the Riemann problem is now obtained by piecing together the solutions of the N Riemann problems (ω_{k-1},ω_k) on different sectors of the (t,x)-plane. The strict separation of the λ_k assures that no overlapping can occur.

In the general case (here and in the rest of the thesis, by *general case* we mean that no assumption on F is made besides strict hyperbolicity) the solution to the Riemann problem (u^L,u^R) was obtained by Stefano Bianchini and Alberto Bressan in [**BB05**]. They first construct, for any left state u^L and for any family $k = 1,\dots,n$, a curve $s \to T^k_s u^L$ of *admissible right states*, defined for $s \in \mathbb{R}$ small enough, such that the Riemann problem $(u^L, T^k_s u^L)$ can be solved by (countable many) admissible shocks (in the sense of limit of viscous traveling profiles), contact discontinuities and rarefactions waves. Then, as in the GNL/LD case, the global solution of (u^L,u^R) is obtained by piecing together the solutions of

N Riemann problems, one for each family: $u^R = T^n_{s_n} \circ \cdots \circ T^1_{s_1} u^L$. (Here some overlapping among shocks and contact discontinuities can occur).

Glimm's and Liu's existence result. The first result about existence of solutions to the general Cauchy problem (1), (2) can be found in the celebrated paper by James Glimm [Gli65] in 1965, in which the existence of solutions is proved again under the assumption that each characteristic field is either GNL or LD. In [Gli65], for any $\varepsilon > 0$ an approximate solution $u^\varepsilon(t,x)$ is constructed by recursion as follows. First of all we can always take (possibly after a linear change of variable in the (t,x)-plane)

$$\lambda_k(u) \in [0,1] \qquad \text{for every } k \text{ and for every } u. \tag{5}$$

Consider now any sampling sequence $\{\vartheta_i\}_{i\in\mathbb{N}} \subseteq [0,1]$. Glimm's algorithm starts by choosing, at time $t=0$, an approximation $\bar{u}^\varepsilon$ of the initial datum $\bar{u}$, such that $\bar{u}^\varepsilon$ is compactly supported, right continuous, piecewise constant with jumps located at point $t = m\varepsilon$, $m \in \mathbb{Z}$. We can thus separately solve the Riemann problems located at $(t,x) = (0, m\varepsilon)$, $m \in \mathbb{Z}$. Thanks to (5), the solution $u^\varepsilon(t,x)$ can now be prolonged up to time $t = \varepsilon$. At $t = \varepsilon$ a restarting procedure is used. The value of u^ε at time ε is redefined as

$$u^\varepsilon(\varepsilon+, x) := u^\varepsilon(\varepsilon-, m\varepsilon + \vartheta_1\varepsilon), \qquad \text{if } x \in [m\varepsilon, (m+1)\varepsilon). \tag{6}$$

The solution $u^\varepsilon(\varepsilon, \cdot)$ is now again piecewise constant, with discontinuities on points of the form $x = m\varepsilon$, $m \in \mathbb{Z}$. If the sizes of the jumps are sufficiently small, we can again solve the Riemann problem at each point $(t,x) = (\varepsilon, m\varepsilon)$, $m \in \mathbb{Z}$ and thus prolong the solution up to time 2ε, where again the restarting procedure (6) is used, with ϑ_2 instead of ϑ_1. The above procedure can be repeated on any time interval $[i\varepsilon, (i+1)\varepsilon]$, $i \in \mathbb{N}$, as far as the size of the jump at each point $(i\varepsilon, m\varepsilon)$, $i \in \mathbb{N}, m \in \mathbb{Z}$, remains small enough, or, in other words, as far as

$$\text{Tot.Var.}(u^\varepsilon(t); \mathbb{R}) \ll 1. \tag{7}$$

In order to prove (7), Glimm introduces a uniformly bounded decreasing functional (also called potential) $t \mapsto Q^{\text{Glimm}}(t) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u})^2$, with the property that at any time $i\varepsilon$, $i \in \mathbb{N}$,

$$\text{Tot.Var.}(u^\varepsilon(i\varepsilon+); \mathbb{R}) - \text{Tot.Var.}(u(i\varepsilon-); \mathbb{R}) \leq \mathcal{O}(1)\big(Q^{\text{Glimm}}(i\varepsilon-) - Q^{\text{Glimm}}(i\varepsilon+)\big). \tag{8}$$

Here and in the following $\mathcal{O}(1)$ denotes a constant which depends only on the flux F. As an immediate consequence, we get $\text{Tot.Var.}(u^\varepsilon(t); \mathbb{R}) \leq \mathcal{O}(1)\text{Tot.Var.}(u^\varepsilon(0); \mathbb{R}) \ll 1$ and thus the solution $u^\varepsilon(t,x)$ can be defined on the whole (t,x)-plane $[0,\infty) \times \mathbb{R}$. The uniform bound on the $\text{Tot.Var.}(u^\varepsilon(t); \mathbb{R})$ yields a compactness on the family $\{u^\varepsilon\}_\varepsilon$: we can thus extract a converging subsequence, which turns out to be, for almost every sampling sequence $\{\vartheta_i\}_i$, a weak admissible solution of the Cauchy problem (1), (2).

Starting from Glimm's pioneering work, finding out suitable decreasing potentials to get a priori bounds on the solutions of the Cauchy problem (1), (2) has been one of the most important directions in the development of the mathematical theory of conservation laws.

Glimm's results was improved in 1977 by Tai Ping Liu in [Liu77], where the author shows that if the sampling sequence $\{\vartheta_i\}$ is *equidistributed*, that means that for any $\lambda \in [0,1]$,

$$\lim_{j\to\infty} \frac{\text{card}\{i \in \mathbb{N} \mid 1 \leq i \leq j \text{ and } \vartheta_i \in [0,\lambda]\}}{j} = \lambda, \tag{9}$$

then the subsequence extracted from $\{u^\varepsilon\}_\varepsilon$ converges to a weak admissible solution of (1), (2), thus getting a *deterministic version of the Glimm scheme*. The main novelty in Liu's paper is the construction of a *wave tracing algorithm* which splits each wavefront in the approximate solution into a finite number of discrete *waves* such that the trajectory of each wave can be traced in time and the sum over all waves of the variation of the speed of each single wave in a given time interval $[t_1, t_2]$ is bounded by the decrease of Q^{Glimm} in $[t_1, t_2]$.

The wave-front tracking method. An alternative method for constructing solutions of the Cauchy problem (1), (2), again as a limit of a sequence of piecewise constant approximations, is the *wavefront tracking algorithm*, introduced by Constantine Dafermos [**Daf72**] for scalar equations and Ronald Di Perna [**DiP76**] for 2×2 systems, then extended in [**BJ98, Bre92, Ris93**] to $N \times N$ systems with GNL or LD characteristic fields. The wavefront tracking algorithm starts at time $t = 0$ by taking, as in the Glimm scheme, a piecewise constant approximation of the initial data. The resulting Riemann problems are then solved within the class of piecewise constant functions by using an approximate Riemann solver that replaces centered rarefaction waves with rarefaction fans containing several small jumps traveling with a speed close to the characteristic speed. This approximate solution can now be prolonged until a time t_1 is reached, when two (or more) wavefronts starting from $t = 0$ interact. Since $u(t_1, \cdot)$ is still piecewise constant, the corresponding Riemann problems can again be approximately solved within the class of piecewise constant functions. The solution u can thus be prolonged up to a time t_2 when, again, two wavefronts collide, and so one.
In using front tracking approximations to prove existence of the Cauchy problem (1), (2), the two main difficulties derive from the fact that the number of lines of discontinuity may approach infinity in finite time and the total variation of the solution can blow up. As in the Glimm scheme, one of the fundamental tools to overcome such difficulties is again the Glimm potential Q^{Glimm}.

The semigroup approach. A different line of research, related to the analysis of uniqueness and stability issues, led to the introduction of the notion of *standard Riemann semigroup*.

Definition 2. A *standard Riemann semigroup* for the system of conservation laws (1) is a map $S : \mathcal{D} \times [0, \infty) \to \mathcal{D}$, defined on a domain $\mathcal{D} \subseteq L^1(\mathbb{R}; \mathbb{R}^N)$ containing all functions with sufficiently small total variation, with the following properties:

(1) for some Lipschitz constants L, L',

$$\|S_t\bar{u} - S_s\bar{v}\|_1 \leq L\|\bar{u} - \bar{v}\|_1 + L'|t - s|, \quad \text{for any } \bar{u}, \bar{v} \in \mathcal{D}, \quad t, s \geq 0; \tag{10}$$

(2) if $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small $S_t\bar{u}$ coincides with the solution of (1), (2), which is obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

In the GNL/LD case it is proved (see, among others, [**BCP00**], [**LY99**], [**BLY99**]) that any system of conservation laws admits a standard Riemann semigroup and that at any time $t \geq 0$ the solution $u(t)$ obtained as limit of Glimm approximations $u^\varepsilon(t)$ with the initial datum $\bar{u}$, coincides with the semigroup $S_t\bar{u}$. Also in the semigroup approach, one of the most important techniques used to prove stability results is the construction of suitable decreasing potential defined on pair of solutions (see [**BLY99**]).

Convergence rate of the Glimm scheme. Relying on the existence of the standard Riemann semigroup for GNL/LD systems, in 1998 A. Bressan and Andrea Marson further improved the theory of Glimm's sampling method. They show in [**BM98**] that, if the sampling sequence $\{\vartheta_i\}$, satisfies the additional assumption

$$\sup_{\lambda \in [0,1]} \left| \lambda - \frac{\operatorname{card}\{i \in \mathbb{N} \mid j_1 \leq i < j_2 \text{ and } \vartheta_i \in [0, \lambda]\}}{j_2 - j_1} \right| \leq C \cdot \frac{1 + \log(j_2 - j_1)}{j_2 - j_1}. \tag{11}$$

(and it is not difficult to prove that such a sequence exists), then the Glimm approximate solutions $u^\varepsilon(T)$ converges to the exact weak admissible solution $u(T) = S_T\bar{u}$; more precisely, the following limit holds:

$$\lim_{\varepsilon\to 0} \frac{\big\|u^\varepsilon(T,\cdot) - S_T\bar{u}\big\|_{L^1}}{\big|\log\varepsilon\big|\sqrt{\varepsilon}} = 0. \tag{12}$$

The technique used in [BM98] to prove (12) is as follows. Thanks to the Lipschitz property of the semigroup (10), in order to estimate the distance

$$\big\|u^\varepsilon(T,\cdot) - u(T,\cdot)\big\|_{L^1} = \big\|u^\varepsilon(T,\cdot) - S_T\bar{u}\big\|_{L^1},$$

we can partition the time interval $[0,T]$ in subintervals $J_r := [t_r, t_{r+1}]$ and estimate the error

$$\|u^\varepsilon(t_{r+1}) - S_{t_{r+1}-t_r}u^\varepsilon(t_r)\|_{L^1} \tag{13}$$

on each interval J_r. The error (25) on J_r comes from two different sources:

(1) first of all there is an error due to the fact that in a Glimm approximate solution, roughly speaking, we give each wavefront either speed 0 or speed 1 (according to the sampling sequence $\{\vartheta_i\}_i$), while in the exact solution it would have a speed in $[0,1]$, but not necessarily equal to 0 or 1;

(2) secondly, there is an error due to the fact that some wavefronts can be created at times $t > t_r$, some wavefronts can be canceled at times $t < t_{r+1}$ and, above all, some wavefronts, which are present both at time t_r and at time t_{r+1}, can change their speeds, when they interact with other wavefronts.

The first error source is estimated by choosing the intervals J_r sufficiently large in order to use estimate (11) with $j_2 - j_1 \gg 1$. The second error source can be estimated (choosing the intervals J_r not too large) using the bound on the change in speed of the waves present in the approximate solution provided, in the GNL/LD case, by Liu in [Liu77] through his wave tracing algorithm and the Glimm potential Q^{Glimm}.
As $\varepsilon \to 0$, it is convenient to choose the asymptotic size of the intervals J_r in such a way that the errors in (1) and (2) have approximately the same order of magnitude. In particular, the estimate (12) is obtained by choosing $|J_r| \approx \sqrt{\varepsilon}\log|\log\varepsilon|$.

Results in the non-convex setting. Up to now, all the results we presented were obtained under the assumption that each characteristic field is either GNL or LD. We consider now the general case, when this "convexity" assumption is removed and the only property of F is its strict hyperbolicity (3).

The problem of finding a suitable decreasing potential to bound the increase of $t \mapsto \text{Tot.Var.}(u^\varepsilon(t);\mathbb{R})$ for a Glimm approximate solution u^ε (see (8)) was solved first by Tai Ping Liu in [Liu81] for fluxes with a finite number of inflection points. Later, in [Bia03], Bianchini solved the problem for general hyperbolic fluxes, introducing the cubic functional

$$t \mapsto Q^{\text{cubic}}(t) := \iint |\sigma(t,s) - \sigma(t,s')|\,ds\,ds' \approx \mathcal{O}(1)\text{Tot.Var.}(u^\varepsilon(t))^3, \tag{14}$$

where s, s' are two waves in the approximate solution at time t and $\sigma(t,s), \sigma(t,s')$ denote their speed.

In [BB05] Bianchini and Bressan also proved the following fundament theorem, which provides existence, uniqueness and stability of the solutions to (1), (2).

Theorem 1. *Any strictly hyperbolic F admits a standard Riemann semigroup $\{S_t \,|t \geq 0\}$ of* vanishing viscosity solutions *with small total variation obtained as the (unique) limits of solutions to the viscous parabolic approximations*

$$u_t + F(u)_x = \mu u_{xx}, \tag{15}$$

when the viscosity $\mu \to 0$. *The semigroup* S *is defined on*

$$\mathcal{D} := \left\{ u \in L^1(\mathbb{R};\mathbb{R}^N) \;\middle|\; \text{Tot.Var.}(u) \ll 1,\ \lim_{x\to-\infty} u(x) \in K \right\}$$

and satisfies the Lipschitz condition

$$\|S_t\bar{u} - S_s\bar{v}\|_1 \le L\|\bar{u} - \bar{v}\|_1 + L'|t-s|, \quad \textit{for any } \bar{u},\bar{v} \in \mathcal{D}, \quad t,s \ge 0. \tag{16}$$

As we have already pointed out, Theorem 1 is the most comprehensive result about existence, uniqueness and stability of the solutions of the Cauchy problem (1), (2). Its proof, however, relies on the deep analysis of the viscous approximations (15) through parabolic techniques. It seemed thus worthwhile to develop a *purely hyperbolic* theory (based, for instance, on Glimm or wavefront tracking approximations) to prove existence, uniqueness and stability results in the general case, when no assumption on F is made except the strict hyperbolicity. The reason of the opportunity of developing such purely hyperbolic theory, other than theoretical interest, can be mainly found in the fact that the hyperbolic approximate solutions (like Glimm's ones or wavefront tracking ones) are piecewise constant functions with discontinuities traveling on a finite number of straight line and they thus can be visualized, analyzed and used much more easily than the parabolic approximations which solve (15).

The construction of wavefront tracking approximations in the general setting and the proof of their convergence to the semigroup solution $S_t\bar{u}$ provided by Theorem 1 was performed by Fabio Ancona and A. Marson in [AM07]. In a very similar way to the GNL/LD case, also in the general case the main difficulties come from the fact that the number of lines of discontinuity may approach infinity in finite time and the total variation of the solution can blow up. The cubic functional Q^{cubic} defined in (14) is sufficiently sharp to be used to overcome such difficulties.

The analysis of the Glimm scheme presents more difficulties. It is not difficult to show (see for instance Theorem 2.16) that, as for wavefront solutions, the cubic functional Q^{cubic} is sharp enough to construct, for any $\varepsilon > 0$ a Glimm approximation u^ε defined for all times $t \in [0,+\infty)$. However (as pointed out in [AM11b], see also [HJY10] and [HY10])) the proof of the convergence of the Glimm scheme in the deterministic setting and the proof of estimate (12) on the convergence rate can not be achieved now through the cubic functional Q^{cubic}, while, on the contrary, in the GNL/LD setting those results were obtained through the same functional Q^{Glimm} used to bound the total variation of the solution.

Indeed, as observed by Ancona and Marson in [AM11b], the sum over all waves of the variation of their speed in a time interval $[t_1,t_2]$ (which, as in the GNL/LD case, is the crucial term to estimate) is a *quadratic* quantity and thus can not be estimated through a cubic functional. The following example can clarify the problem.

Consider a scalar equation ($N=1$) and consider an interaction between two positive shocks of strength, respectively, s_1, s_2 and speed σ_1, σ_2. By the well-known properties of the scalar equation, after the collision a single shock is present, whose strength is s_1+s_2 and whose speed is

$$\sigma = \frac{\sigma' s' + \sigma'' s''}{s' + s''}.$$

Therefore, in this simple example, the (weighted) sum of the variation of speed of the wavefronts is

$$|\sigma - \sigma'||s'| + |\sigma - \sigma''||s''| = \frac{|\sigma' - \sigma''|}{s' + s''}|s'||s''| \le \|F''\|_\infty |s'||s''|, \tag{17}$$

and it is evident from the last inequality that such variation of speed is (in the worst case) quadratic w.r.t. the total variation of the shocks involved in the interaction. In order to prove the convergence of the deterministic version of the Glimm scheme and a sharp rate of convergence like the one in (12) it is thus necessary to introduce some new quadratic interaction potential. This is exactly the point where our research started three years ago.

Overview of our contributions

Aim of the remaining part of this introduction is to answer the following questions: *what* are our main contributions and *why* we decided to study such problems; *which* line we followed in our researches and *why* we decided to follow this line; in other words, which intermediate steps we considered before getting the final results and how the three results we present here have been developed one from each other.

A quadratic interaction estimate. As mentioned at the end of the first part of the Introduction, our research started with the study of the papers [AM11b], [HJY10], [HY10], where a quadratic interaction estimate is discussed. As in (17), such quadratic estimate can be easily explained in the case of a wavefront tracking solution to (1), (2) in the scalar case $N = 1$. Let $\{t_j\}_{j=1,\dots,J}$ be the times at which two (or more) wavefronts having the same sign interact. We assume for simplicity that at time time t_j only two wavefronts interact. The quadratic interaction estimate can be written as

$$\sum_{t_j \text{ interaction}} \frac{|\sigma(s'_j) - \sigma(s''_j)||s'_j||s''_j|}{|s'_j| + |s''_j|} \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u})^2. \tag{18}$$

In the above formula s'_j, s''_j are the wavefronts which interact at time t_j, $\sigma(s'_j)$ (resp. $\sigma(s''_j)$) is the speed of the wavefront s'_j (resp. s''_j) and $|s'_j|$ (resp. $|s''_j|$) is its strength. Notice that, by (17), the above estimate is exactly the global change in speed of the wavefront present in the solution due to interactions between wavefronts with the same sign (the interactions between wavefronts of opposite sign are much less complicated to study and thus we do not consider them for the moment). As it is shown by some counterexamples in [AM11a, BM14a], some points in the proofs of (17) presented in the papers [AM11b], [HJY10], [HY10] contain, in our opinion, some gaps, which justified the publication of a new and different proof in [BM14a], [BM14b] and [BM15b]: this is the first result of this thesis. The precise statement of the theorem (which is the generalization of estimate (18) to a Glimm approximate solution in the vector case) requires the introduction of further notations. For this reason we think that is is more convenient first to discuss the line of research we followed in the three cited papers to prove (the generalization of) estimate (18) and then to present the precise statement of the theorem at the end of this section, see Theorem A below. The proof of Theorem A is the topic of Chapter 3.

In all the three papers [BM14a], [BM14b], [BM15b], the proof of estimate (18) relies on two main tools:

(a) a new wave tracing algorithm, which, in the same spirit as [Liu77], splits each wavefront in the approximate solution into a (discrete or continuous) set of elementary pieces called *waves*; more precisely, we introduce a map $\mathtt{x}(t, w)$, called *the position map*, which gives the position of every wave w at every time t and three quantities $\mathcal{S}(w), \mathtt{t}^{\mathrm{cr}}(w), \mathtt{t}^{\mathrm{canc}}(w')$ which correspond to the *sign* ± 1 of a given wave w, to its *creation time* and to its *cancellation time*;

(b) a new interaction functional

$$t \mapsto \mathfrak{Q}(t) := \iint_{\{(w,w') \text{ pair of waves}\}} \mathfrak{q}(t, w, w') dw dw', \tag{19}$$

where $\mathfrak{q}(t, w, w')$ is a quantity called *the weight of the pair of waves* (w, w') *at time* t; the main features of $\mathfrak{Q}$ are that it has bounded variation and its decrease at each interaction time t_j controls the quantity

$$\frac{|\sigma(s'_j) - \sigma(s''_j)||s'_j||s''_j|}{|s'_j| + |s''_j|}, \tag{20}$$

which is the quantity to be summed on the l.h.s. of (18).

We remark that the functional $\mathfrak{Q}$ is the natural extension of the original Glimm functional Q^{Glimm}; indeed $\mathfrak{Q}$ reduces to Q^{Glimm} when a GNL/LD system is considered and the weight $\mathfrak{q}(t, w, w')$ is defined to be 1 if w, w' have different positions at time t (i.e. $\mathtt{x}(t, w) < \mathtt{x}(t, w')$) or 0 if they have the same position, ($\mathtt{x}(t, w) = \mathtt{x}(t, w')$).

We decided to study estimate (18) first of all in the most simple situation: a wavefront tracking solution to the scalar equation (1), $N = 1$. This has been done in [BM14a]. The advantage of considering first the scalar case relies on the fact that all the wavefronts belong to the same family. Therefore the collisions among wavefronts can be only *interactions* among wavefronts having the same sign, or *cancellations*, i.e. collisions among wavefronts having opposite sign. Nevertheless, even in the scalar situation the analysis is already quite complicated. Indeed, the main problem in proving (18) in the non-convex setting (which is the scalar counterpart of the lack of the GNL condition) is the following. If F is strictly convex, when a cancellation occurs, i.e. a shock meets a rarefaction, the rarefaction and a part of the shock are canceled, while the outgoing Riemann problem is made by a single shock. Therefore, it is not difficult to see that Q^{Glimm} is decreasing. Indeed, two waves which have the same position $\mathtt{x}(\bar{t}, w) = \mathtt{x}(\bar{t}, w')$ at some time $\bar{t}$ must have the same position $\mathtt{x}(t, w) = \mathtt{x}(t, w')$ for any time $t \in [\bar{t}, \min\{\mathtt{t}^{\text{canc}}(w), \mathtt{t}^{\text{canc}}(w')\})$ until one of the two is canceled. If, on the contrary, F is not convex, a shock which collides with a rarefaction can be split in several pieces and thus Q^{Glimm} is not decreasing any more. Our idea to solve this problem was to associate to each pair of waves (w, w') a *characteristic interval* $\mathcal{I}(t, w, w')$ which summarize the past history of the two waves from the *time of their last splitting* $\mathtt{t}^{\text{split}}(t, w, w')$ and to assign to each pair of waves a positive weight defined, roughly speaking, as

$$\mathfrak{q}(t, w, w') \approx \frac{\substack{\text{difference in speed of } w, w' \text{ for the Riemann problem in } \mathcal{I}(t, w, w') \\ \text{with the flux } F}}{\text{length of the interval } \mathcal{I}(t, w, w')}.$$

The choice of the weights $\mathfrak{q}(t, w, w')$ is sharp enough to guarantee that, on one side, their positive total variation in time is uniformly bounded and, on the other side, at each interaction they are huge enough to bound (20). The most important conclusion of the analysis in [BM14a] is that the weights $\mathfrak{q}(t, w, w')$ (and thus also the potential $\mathfrak{Q}$) are non-local in time, a situation very different from the standard Glimm interaction analysis of hyperbolic systems of conservation laws.

Since estimate (18) raised from the problem of the convergence of the deterministic version of the Glimm scheme, in paper [BM14a] we also considered an approximate solution to the scalar system $N = 1$ obtained through the Glimm scheme. The main observation in the scalar Glimm case is that the analysis is quite similar to the one performed for the wavefront solutions; however, due to the presence of rarefactions, which are not approximated through a finite number of discontinuities, it is much more convenient to consider a *continuous* wave tracing, differently from the discrete wave tracing algorithm proposed in the work of Liu [Liu77] and considered also in [AM11b], [HJY10]. The choice of a continuous wave tracing will turn out to be very convenient in the analysis of the system case and, in particular, in the construction of the Lagrangian representation, which is the third result of this thesis.

After the scalar case, in [BM14b] we study how the same estimate (18) can be proved in the presence of waves of different families. To this aim, the most simple situation is considered, namely a wavefront solution to the Temple-class triangular system (see [Tem83] for the definition of *Temple class systems*)

$$\begin{cases} u_t + \tilde{f}(u,v)_x = 0, \\ v_t - v_x = 0, \end{cases}$$

with $\frac{\partial \tilde{f}}{\partial u} > -1$, so that local uniform hyperbolicity is satisfied. Besides interactions of waves of the same family and same sign and cancellations, we deal here also with *transversal interactions*, i.e. interactions of waves of different family. The main difficulties here, w.r.t. the scalar case, are the following: a shock can be split not only by a cancellation, but also by a transversal interaction; at any given time t, the reduced flux (see (2.8) for the definition of *reduced flux*) of a pair of waves (w,w') at the time $\mathtt{t}^{\text{split}}(t,w,w')$ of their last splitting before t can be different from the reduced flux of the same pair of waves at *the time* $\mathtt{t}^{\text{int}}(t,w,w')$ *of their next interaction* after t. These problems are solved in [BM14b] through the definition of an *effective flux function* $\mathtt{f}^{\text{eff}}(t)$, depending on time, which contains all the information about the "convexity/concavity" of each characteristic family and the introduction of a *partition* $\mathcal{P}(t,w,w')$ of the characteristic interval $\mathcal{I}(t,w,w')$. Roughly speaking, the new definition of the weights becomes, in this case,

$$\mathfrak{q}(t,w,w') \approx \frac{\substack{\text{difference in speed of } w,w' \text{ for the Riemann problem in } \mathcal{I}(t,w,w') \\ \text{w.r.t the partition } \mathcal{P}(t,w,w') \text{ and the flux } \mathtt{f}^{\text{eff}}(t)}}{\text{length of the interval } \mathcal{I}(t,w,w')}.$$

Finally in [BM15b] we prove estimate (18) for a general system of conservation laws (1), without any assumption on F except the strict hyperbolicity (3). The two main difficulties in the general case are the following. First, the Riemann problems in the general case are not solved anymore taking the convex/concave envelope of a given flux (as it happens in the scalar case and also in the triangular system when Riemann coordinates are used), but they are solved through the solution of a fixed point problem in the space of curves $\gamma = (u, v_k, \sigma_k)$ (where k is a fixed family in $\{1,\dots,N\}$) equipped with a suitable norm, as proved for the first time in [BB05]. However, the norm in [BB05] is not sharp enough to estimate the change in speed of the waves, which is the quantity we are interested in. Therefore we were forced to introduce a new distance between a pair of curves $\gamma = (u, v_k, \sigma_k)$, $\gamma' = (u', v'_k, \sigma'_k)$ in which the term $\|\sigma_k - \sigma'_k\|_1$ plays a crucial role. The second difficulty in the general case is that the weight used for the analysis of the triangular system is a map in $L^1(dwdw')$ and not in $L^\infty(dwdw')$ and for this reason it is very sensitive to small cancellations. This problem, which does not appear in the simplified setting considered in [BM14b], is a main source of troubles in the general case. Its solution is obtained adapting the definitions of $\mathcal{I}(t,w,w')$ and of the associated partition $\mathcal{P}(t,w,w')$ in order to include information not only about the past history of the pair (w,w') from $\mathtt{t}^{\text{split}}(t,w,w')$, but also about the future history of (w,w') up to the time of their next interaction $\mathtt{t}^{\text{int}}(t,w,w')$. As a last remark about paper [BM15b], we observe that in this paper only the analysis for the Glimm scheme is performed; the reason of this choice is, first of all, that one of the applications of estimate (18) concerns the convergence rate of the Glimm scheme; as a second motivation, in this case, both estimate (18) and its proof can be written in a very clean form, since from each grid point $(i\varepsilon, m\varepsilon)$, $i \in \mathbb{N}$, $m \in \mathbb{Z}$, an exact (not approximate) solution to a Riemann problem arises, while in the wavefront tracking we should have dealt with several different approximate Riemann solvers.

In this thesis we decided to present, in Chapter 3, the proof of estimate (18) and the construction of the functional $\mathfrak{Q}$ directly for a Glimm approximate solution to the general strictly hyperbolic $N \times N$ case, i.e. in the situation considered in [BM15b] (even with some modifications in the definition of the partitions and the weights, as we will see later). A simplified proof (for a less general system) can be found in [Mod15b].

We conclude this section with the precise statement of the theorem we will prove in Chapter 3, which require the introduction of some notation. Let (u^L, u^M), (u^M, u^R) be two Riemann problems with a common state u^M, and consider the Riemann problem (u^L, u^R). As we have already observed, it is shown in [BB05] (and it will be proved also in Section 2.1) that if $|u^M - u^L|, |u^R - u^M| \ll 1$, then one can solve the three Riemann problems as follows:

$$u^M = T^N_{s'_N} \circ \cdots \circ T^1_{s'_1} u^L, \qquad u^R = T^N_{s''_N} \circ \cdots \circ T^1_{s''_1} u^M, \qquad u^R = T^N_{s_N} \circ \cdots \circ T^1_{s_1} u^L,$$

where for each $k = 1, \dots, N$, $s'_k, s''_k, s_k \in \mathbb{R}$ and $(s, u) \mapsto T^k_s u$ is the map which at each left state u associates the right state $T^k_s u$ such that the Riemann problem $(u, T^k_s u)$ has an entropy admissible solution made only by wavefronts with total strength $|s|$ belonging to the k-th family (see page XII in this introduction).

We are interested in studying how much the speed of the wavefronts of the two incoming Riemann problems can change after the collision. More precisely, for each family k, writing for brevity

$$\mathbf{I}(s) = \big[\min\{s, 0\}, \max\{s, 0\}\big] \setminus \{0\},$$

let us denote by

$\sigma'_k : \mathbf{I}(s_k) \to (\hat{\lambda}_{k-1}, \hat{\lambda}_k)$	the speed function of the wavefronts of the k-th family for the Riemann problem (u^L, u^M),
$\sigma''_k : s'_k + \mathbf{I}(s''_k) \to (\hat{\lambda}_{k-1}, \hat{\lambda}_k)$	the speed function of the wavefronts of the k-th family for the Riemann problem (u^M, u^R),
$\sigma_k : \mathbf{I}(s_k) \to (\hat{\lambda}_{k-1}, \hat{\lambda}_k)$	the speed function of the wavefronts of the k-th family for the Riemann problem (u^L, u^R).

Notice that we are assuming that σ''_k is defined on $s'_k + \mathbf{I}(s''_k)$ instead of $\mathbf{I}(s''_k)$ and that $\hat{\lambda}_{k-1}$, $\hat{\lambda}_k$ are respectively a lower and upper bound for the k-th eigenvalue $\lambda_k(u)$. Let us consider now the L^1-norm of the speed difference between the waves of the Riemann problems (u_L, u_M), (u_M, u_R) and the outgoing waves of (u_L, u_R):

$$\Delta\sigma_k(u^L, u^M, u^R) := \begin{cases} \big\|(\sigma'_k \cup \sigma''_k) - \sigma_k\big\|_{L^1(\mathbf{I}(s'_k + s''_k) \cap \mathbf{I}(s_k))} & \text{if } s'_k s''_k \geq 0, \\ \big\|(\sigma'_k \vartriangle \sigma''_k) - \sigma_k\big\|_{L^1(\mathbf{I}(s'_k + s''_k) \cap \mathbf{I}(s_k))} & \text{if } s'_k s''_k < 0, \end{cases}$$

where $\sigma'_k \cup \sigma''_k$ is the function obtained by piecing together σ'_k, σ''_k, while $\sigma'_k \vartriangle \sigma''_k$ is the restriction of σ'_k to $\mathbf{I}(s'_k + s''_k)$ if $|s'_k| \geq |s''_k|$ or $\sigma''_k \llcorner_{\mathbf{I}(s'_k + s''_k)}$ in the other case, see formulas (1.2), (1.3).

Now consider a right continuous ε-approximate solution constructed by the Glimm scheme (see page XIII or Section 2.3); by simplicity, for any grid point $(i\varepsilon, m\varepsilon)$ denote by

$$\Delta\sigma_k(i\varepsilon, m\varepsilon) := \Delta\sigma_k(u^{i,m-1}, u^{i-1,m-1}, u^{i,m})$$

the change in speed of the k-th wavefronts at the grid point $(i\varepsilon, m\varepsilon)$ arriving from points $(i\varepsilon, (m-1)\varepsilon)$, $((i-1)\varepsilon, (m-1)\varepsilon)$, where $u^{j,r} := u(j\varepsilon, r\varepsilon)$. The first result of this thesis is that the sum over all grid points of the change in speed is bounded by a quantity which depends only on the flux F and the total variation of the initial datum and does not depend on ε. More precisely, the theorem we will prove in Chapter 3 is the following.

THEOREM A. *It holds*

$$\sum_{i=1}^{+\infty} \sum_{m \in \mathbb{Z}} \Delta\sigma_k(i\varepsilon, m\varepsilon) \leq \mathcal{O}(1) \text{Tot.Var.}(\bar{u}; \mathbb{R})^2, \tag{21}$$

where $\mathcal{O}(1)$ is a quantity which depends only on the flux F.

We explicitly notice that $\Delta\sigma_k$ is exactly the *variation of the speed of the waves when joining two Riemann problems* and, in the case of an interaction between two shocks of the same family and having the same sign, it corresponds exactly to the quantity to be summed on the l.h.s. of (18). Estimate (21) is thus exactly the generalization of estimate (18) to the case of a Glimm approximate solution to the $N \times N$ system.

As a final remark of this section, we observe that estimate (21) has a natural extension in the viscous setting, when a viscosity term μu_{xx} is added on the l.h.s. of (1), see (15). While the building block of the solution of the hyperbolic equation (1) is the Riemann problem, in the viscous case the building block is the *viscous traveling profile*, i.e. a solution of the form

$$u(t,x) = U(x - \lambda t) \tag{22}$$

which satisfies the second order ODE

$$U'' = (Df(U) - \lambda)U'.$$

In this case the velocity of the viscous profile is λ and it holds

$$\lambda = -\frac{u_t}{u_x}.$$

In the inequality (21) the l.h.s. is the sum over all grid points $(i\varepsilon, m\varepsilon)$, $i \in \mathbb{N}$, $m \in \mathbb{Z}$ on the (t,x) plane of the change in speed of the wavefronts present at point $(i\varepsilon, m\varepsilon)$ multiplied by their strength. Hence its equivalent in the viscous setting is the integral over the (t,x) plane of the change in speed multiplied by the strength of the viscous profile at point (t,x), i.e.

$$\iint_{[0,\infty)\times\mathbb{R}} \left|\partial_t \frac{u_t}{u_x}\right| |u_x| dxdt = \iint_{[0,\infty)\times\mathbb{R}} \left|u_{tt} - \frac{u_t}{u_x} u_{tx}\right| dxdt. \tag{23}$$

It would thus be nice to obtain an uniform estimate of (23) in terms of the total variation of the initial datum. However, though the term to estimate can be written down in a very clean form (23), it is not clear at all which technique can be used in order to estimate it. Indeed, the method we use to prove (21) relies heavily on the wave tracing algorithm we developed, in particular on the notion of *position* $\mathtt{x}(t,w)$ *of a given wave* w *at a given time* t: in the viscous case it is not clear any more what this concept means, since the traveling profiles (22) are not localized in space, and thus it is not at all obvious how to define a suitable functional which can play the role that $\mathfrak{Q}$ has in the hyperbolic case.

Sharp convergence rate of the Glimm scheme. We have already pointed out that estimate (18) comes out from the analysis of the convergence rate of the Glimm scheme, where a functional which bounds the change in speed of the elementary waves in the approximate solution is needed. Therefore, after proving estimate (18) in the three cited papers [BM14a], [BM14b], [BM15b], we explicitly proved in [MB15] that the same estimate on the rate of convergence, obtained by Bressan and Marson in [BM98] for a GNL/LD system, holds also in the general strictly hyperbolic setting. More precisely, we proved the following theorem.

THEOREM B. *Consider the Cauchy problem* (1)-(2) *and assume that the system* (1) *is strictly hyperbolic. Let* u^ε *be a Glimm approximate solution with mesh size* $\varepsilon > 0$ *and sampling sequence satisfying* (11)*, and denote by* $t \mapsto S_t\bar{u}$ *the semigroup of vanishing viscosity solutions, provided by Theorem* 1*. Then for every fixed time* $T \in [0, +\infty)$ *the following limit holds:*

$$\lim_{\varepsilon\to 0} \frac{\left\|u^\varepsilon(T,\cdot) - S_T\bar{u}\right\|_1}{\sqrt{\varepsilon}|\log\varepsilon|} = 0. \tag{24}$$

In proving (24) we discovered the following phenomenon. The quadratic estimate (18) is necessary and sufficient in the GNL/LD case to prove both the convergence of the deterministic version of the Glimm scheme [Liu77] and estimate (24) on its rate of convergence. In the general setting, estimate (18) is sufficient to extend Liu's proof [Liu77] of the convergence of the deterministic Glimm scheme (even if we will not prove this statement explicitly, we observe that, at the best of our knowledge, no proof of this fact has been proposed in the literature in the general strictly hyperbolic setting). On the contrary, estimate (18) is necessary, but not sufficient, to prove the bound on the convergence rate of the Glimm scheme (24). The problem is the following. As we explained when we introduced the paper [BM98] on the convergence rate of the Glimm scheme in the GNL/LD setting (page XIV), the crucial point is to partition the time interval $[0,T]$ into subintervals $[t_r, t_{r+1}]$ and to estimate the error

$$\|u^\varepsilon(t_{r+1}) - S_{t_{r+1}-t_r} u^\varepsilon(t_r)\|_{L^1} \tag{25}$$

in each of these intervals, through a suitable decreasing (quadratic) potential. The technique we use in [MB15] to prove (24) is analogous to the one used in [BM98]. We construct a wavefront auxiliary map $\psi : [t_r, t_{r+1}] \times \mathbb{R} \to \mathbb{R}^N$ such that there is a correspondence between waves in u^ε and waves in ψ and, moreover, each wave w has the same initial position at time t_r in u^ε and in ψ and the same final position at time t_{r+1} in u^ε and in ψ. In the GNL/LD, however, if two waves have the same position at t_r and different positions at t_{r+1}, then they must be positive waves (rarefactions) and thus their speed do not depend on the other waves with which they are traveling at a given time t. Therefore, their speed at time t_r is the same in u^ε and in ψ. On the contrary, we have already remarked that in the non-convex setting splittings can occur and thus this property is not true any more. Hence, we have been forced to introduce in [MB15] a sharper version of the potential $\mathfrak{Q}(t)$ and of the weights $\mathtt{q}(t,w,w')$ which takes into account the whole history of the pair (w,w') not only from $\mathtt{t}^{\text{split}}(t,w,w')$ to $\mathtt{t}^{\text{int}}(t,w,w')$ but on a generic time interval $[t_r, t_{r+1}]$.

This sharper version of the potential $\mathfrak{Q}(t)$ is the one we present in this thesis in Chapter 3. The proof of the convergence rate (24) of the Glimm scheme, which is the second result of this thesis, can be found in Chapter 4.

The Lagrangian representation. The last results of this thesis arises from the following observation. To prove the quadratic interaction estimate (18) and to construct the functional $\mathfrak{Q}$ we introduced the notion of *continuous wave tracing* for the (Glimm) approximate solution u^ε, $\varepsilon > 0$, i.e. the *set of waves* $\mathcal{W}^\varepsilon \subseteq \mathbb{R}$ together with the *position map* $\mathtt{x}^\varepsilon(t,w)$, $t \in [0,\infty)$, $w \in \mathcal{W}^\varepsilon$ and the three maps $\mathcal{S}^\varepsilon(w)$, $(\mathtt{t}^{\text{cr}})^\varepsilon(w)$, $(\mathtt{t}^{\text{canc}})^\varepsilon(w)$, respectively the *sign* of w, its *creation time* and its *cancellation time*. We observed that the family, parametrized by $\varepsilon > 0$, of all the wave tracings shares an intrinsic compactness. Indeed the maps $w \mapsto \mathtt{x}^\varepsilon(t,w)$ are increasing for every fixed time t, while the maps $t \mapsto \mathtt{x}^\varepsilon(t,w)$ are Lipschitz for any fixed wave w. Therefore, we can pass to the limit to get a *position map* $\mathtt{x}(t,w)$ such that $\mathtt{x}^\varepsilon(t,\cdot) \to \mathtt{x}(t,\cdot)$ in L^1 as $\varepsilon \to 0$ (see Proposition 5.38). Defining the *density function* as

$$\rho^\varepsilon(t,w) := \mathcal{S}^\varepsilon(w)\chi_{[(\mathtt{t}^{\text{cr}})^\varepsilon(w),(\mathtt{t}^{\text{canc}})^\varepsilon(w))}(t),$$

it is immediate to see that $w \mapsto \rho^\varepsilon(t,w)$ are uniformly bounded in L^∞; moreover, it turns out that the distributional derivative w.r.t. time $D_t\rho^\varepsilon$ is a finite Radon measure on the plane and thus we can pass to the limit also the density functions to get a map $\rho(t,w)$ such that $\rho^\varepsilon(t,\cdot) \to \rho(t,\cdot)$ weakly* in L^∞ for any t, as $\varepsilon \to 0$, and $D_t\rho$ is a finite Radon measure on the plane (see Proposition 5.39). We thus obtain two maps, namely $\mathtt{x}(t,w)$ and $\rho(t,w)$, respectively the *position* at time t of the wave w and the *density* at time t of the wave w, which, in some sense, provide a wave tracing for the exact solution $u(t) = S_t\bar{u}$, where S is the semigroup introduced in Theorem 1. We will call this "limit" wave tracing *Lagrangian representation*. It is explained below the reason of this name.

To better clarify what we mean by Lagrangian representation, we present now the precise definition of Lagrangian representation just in the scalar case $N = 1$ (the vector case would require many further notations, see Definition 5.21), we state the main existence theorem which will be proved in Chapter 5 and we show how the existence of a Lagrangian representation can be proved in three very simple examples.

DEFINITION 3. Let $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be a solution of the Cauchy problem (1), (2) in the case $N = 1$. A *Lagrangian representation* for u up to a fixed time $T > 0$ is a 3-tuple $(\mathcal{W}, \mathtt{x}, \rho)$, where

$$\begin{aligned}
&\mathcal{W} \subseteq \mathbb{R} \text{ is called the } \textit{set of waves},\\
&\mathtt{x} : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is the } \textit{position function},\\
&\rho : [0,T] \times \mathcal{W} \to [-1,1] \text{ is the } \textit{density function},
\end{aligned}$$

and, for every time $t \in [0, T]$ up to a countable set, the following properties hold:

(i) for every fixed time $t \in [0, T]$, $w \mapsto \mathtt{x}(t, w)$ is increasing; for every fixed $w \in \mathcal{W}$, $t \mapsto \mathtt{x}(t, w)$ is 1-Lipschitz and therefore is it differentiable for a.e. time $t \in [0, T]$; moreover

$$\frac{\partial \mathtt{x}}{\partial t}(t, w) = \lambda(t, \mathtt{x}(t, w)), \text{ for } |\rho(\cdot, w)|\mathcal{L}^1\text{-a.e. time } t \in [0, T], \tag{26}$$

where $\lambda = \lambda(t, x)$ is the scalar field given by

$$\lambda(t,x) := \begin{cases} f'(u(t,x)) & \text{if } u(t,\cdot) \text{ is continuous at } x,\\ \frac{f(u(t,x+))-f(u(t,x-))}{u(t,x+)-u(t,x-)} & \text{if } u(t,\cdot) \text{ has a jump at } x; \end{cases} \tag{27}$$

(ii) extending on the whole $\mathbb{R}^2$ the maps ρ to zero outside the set $[0, T] \times \mathcal{W}$, the distribution $D_t\rho$ is a finite Radon measure on $\mathbb{R}^2$;

(iii) the distributional derivative of $u(t, \cdot)$ w.r.t. x satisfies

$$D_x u(t) = \mathtt{x}(t)_\sharp\big(\rho(t)\mathcal{L}^1|_{\mathcal{W}}\big),$$

where $D_x u(t)$ is the distributional derivative (viewed as a measure) of the map $x \mapsto u(t, x)$, $\mathcal{L}^1$ is the Lebesgue measure and $\sharp$ denotes the push-forward of measures.

Point (i) describes the regularity properties of the position map $\mathtt{x}$ and requires that the trajectory $\mathtt{x}(t, w)$ of any fixed wave w is, roughly speaking, a characteristic curve. Point (ii) describes the regularity properties of the map ρ and, in particular, requires that not too many creations/cancellations of waves take place. Finally, Point (iii) requires that the maps $\mathtt{x}, \rho$ are enough to reconstruct the solution u (provided one knows its value as $x \to -\infty$) at any fixed time t.

The existence of a Lagrangian representation (even in the system case) is the third result of this thesis and it is presented in Chapter 5, where we will also provide a precise definition of *Lagrangian representation* for a general solution to the $N \times N$ system (1), (2), see Definition 5.21. The theorem we will prove is the following.

THEOREM C. *Let $u(t) := S_t\bar{u}$ be the vanishing viscosity solution of the $N \times N$ system of conservation laws (1) with initial datum $\bar{u}$. Let $T > 0$ be a fixed time. Then there exists a Lagrangian representation of u up to the time T, which moreover satisfies the following condition: up to countable many times, for every $x \in \mathbb{R}$*

$$x \text{ is a continuity point for } u(t,\cdot) \iff \int_{\mathtt{x}(t)^{-1}(x)} \rho(t,w)dw = 0. \tag{28}$$

We do not want to explain now how the proof of Theorem C is obtained. A sketch of the proof of Theorem C can be found in Section 5.2. On the contrary, we prefer to clarify Definition 3, showing how a Lagrangian representation can be constructed in three very simple, scalar examples.

EXAMPLE 1. Consider a single scalar Riemann problem (u^L, u^R) at $(t,x) = (0,0)$ and assume that $u^L < u^R$ and it is solved by a single entropic shock of strength $|u^R - u^L|$ traveling with speed σ. In this case a possible Lagrangian representation (up to time time $+\infty$) for the solution u is

$$\begin{aligned}
&\mathcal{W} := \big(0, |u^R - u^L|\big], \text{ the set of waves,}\\
&\mathtt{x}(t,w) = \sigma t \text{ for any } t \in (0,\infty] \text{ and } w \in \mathcal{W},\\
&\rho(t,w) = 1 \text{ for any } t \in (0,\infty] \text{ and } w \in \mathcal{W}.
\end{aligned}$$

It is immediate to verify that the Properties (i), (ii), (iii) in the definition of Lagrangian representation are satisfied. Moreover, from this example it is clear that the map $\mathtt{x}$, in some sense, *transports* the derivative of the initial datum, which is a Dirac's delta located in 0 with strength $u^R - u^L$ along a characteristic line.

A complete similar analysis can be done if $u^L > u^R$, just requiring that $\rho(t,w) = -1$ for any t and w.

EXAMPLE 2. The second example concerns a single interaction between two shocks having the same sign. Assume thus the the initial datum is made by a Riemann problem (u^L, u^M) located at $x = -1$ and another Riemann problem (u^M, u^R) located at $x = +1$. Assume that $u^L < u^M < u^R$; the first shock travels with speed σ'; the second shock travels with speed σ''. If $\sigma' > \sigma''$, the two shocks collide at time $\bar{t} := 2/(\sigma' - \sigma'')$ in some point $\bar{x}$. After the collision a single shock of strength $u^R - u^L$ is generated, traveling with speed

$$\sigma = \frac{\sigma'(u^M - u^L) + \sigma''(u^R - u^M)}{u^R - u^L}.$$

A possible Lagrangian representation for this configuration is the following. The set of waves is $\mathcal{W} := (0, u^R - u^L]$. The position map is defined for $w \in (0, u^M - u^L]$ as

$$\mathtt{x}(t,w) = \begin{cases} -1 + \sigma' t & \text{if } t \in [0,\bar{t}],\\ \bar{x} + \sigma t & \text{if } t \in [\bar{t},\infty], \end{cases}$$

while for $w \in (u^M - u^L, u^R - u^L]$ as

$$\mathtt{x}(t,w) = \begin{cases} +1 + \sigma'' t & \text{if } t \in [0,\bar{t}],\\ \bar{x} + \sigma t & \text{if } t \in [\bar{t},\infty]. \end{cases}$$

Finally the density is defined as $\rho(t,w) = 1$ for any t and any w (no wave is canceled). As before, it is easy to see that Properties (i), (ii), (iii) in Definition 3 are satisfied.

EXAMPLE 3. The third example concerns a single interaction between two shocks having opposite sing, namely a *cancellation*. Consider the flux

$$F(u) = \begin{cases} -u^2 + u & \text{if } u \le \frac{3}{4},\\ \frac{1}{4}u^2 - \frac{7}{8}u + \frac{45}{64} & \text{if } u \ge \frac{3}{4}. \end{cases}$$

Define also the three states

$$u^L := 0, \qquad u^M := \frac{5}{4}, \qquad u^R := \frac{3}{4}.$$

Assume that the initial datum $\bar{u}$ is made by the Riemann problem (u^L, u^M) located at $x = -1$ and the Riemann problem (u^M, u^R) located at $x = 1$. It is easy to see that the Riemann problem (u^L, u^M) is solved by a single shock traveling with speed $\sigma' := 0$, while the Riemann problem (u^M, u^R) is solved by a single shock traveling with speed $\sigma'' := -3/8 < 0$. Therefore the two shocks collide at some point $(\bar{t}, \bar{x})$. The outgoing Riemann problem $(u^L, u^R) = (0, 3/4)$ is solved by a single shock traveling with speed $\sigma = 1/4$. In this configuration a possible Lagrangian representation (up to time $+\infty$) is thus:

$$\mathcal{W} := \big(0, 7/4\big].$$

The density map is defined as

$$\rho(t,w) := \begin{cases} 1 & \text{for } w \in (0,3/4] \text{ and any } t, \\ 1 & \text{for } w \in (3/4,5/4] \text{ and } t \in [0,\bar{t}), \\ 0 & \text{for } w \in (3/4,5/4] \text{ and } t \in [\bar{t},\infty), \\ -1 & \text{for } w \in (5/4,7/4] \text{ and } t \in [0,\bar{t}), \\ 0 & \text{for } w \in (5/4,7/4] \text{ and } t \in [\bar{t},\infty), \end{cases}$$

meaning that the positive waves in $(3/4, 5/4]$ and the negative waves in $(3/4, 7/4]$ are canceled at time $\bar{t}$. Finally the position map is defined for $w \in (0, 5/4]$ as

$$\mathtt{x}(t,w) := \begin{cases} -1 + \sigma' t & \text{if } t \in [0,\bar{t}], \\ \bar{x} + \sigma t & \text{if } t \in [\bar{t},\infty], \end{cases}$$

meaning that before the interaction all the waves in $(0, 5/4]$ travel together on a big shock starting at $t = 0$ in $x = -1$ and having speed σ', while, after the collision, they travel on a a big shock having speed σ. For the negative waves $w \in (5/4, 7/4]$, the position is defined as

$$\mathtt{x}(t,w) := \begin{cases} 1 + \sigma'' t & \text{if } t \in [0,\bar{t}], \\ \bar{x} + \sigma t & \text{if } t \in [\bar{t},\infty], \end{cases}$$

meaning that before the collision they travel on the shock starting at $(t, x) = (0, +1)$ and after the collision they travel (as canceled waves) on the shock generated at $(\bar{t}, \bar{x})$ (in general, canceled waves are attached to the last surviving wave, such that $w \mapsto \mathtt{x}(t, w)$ is still increasing). It is not difficult to prove that, also in this case, the properties required in the definition of Lagrangian representation hold.

In the three previous example is was pretty easy to construct explicitly a Lagrangian representation. For a system with an initial datum not so simple, however, the procedure is more complicated and require, as we stressed before, the proof of the convergence of the approximate wave tracing.

We discuss now the reason why we call this "limit" wave tracing a *Lagrangian representation*. The notion of Lagrangian flow is a well-established concept in the theory of the *transport equation* and in the study of the *Euler equation*. For instance, in the linear transport equation

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla_x u(t,x) & = 0, \\ u(t,0) & = \bar{u}(x), \end{cases} \tag{29}$$

where

$$u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}, \text{ is the unknown,}$$
$$b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d \text{ is a given vector field,}$$

the solution to (29) presents a strong connection with the Lagrangian flow $\mathtt{x} : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ generated by the ODE

$$\begin{cases} \dfrac{\partial \mathtt{x}}{\partial t}(t,x) & = b(t,\mathtt{x}(t,x)), \\ \mathtt{x}(0,x) & = x. \end{cases} \tag{30}$$

Similarly, to the incompressible Euler equation

$$\begin{cases} \partial_t u(t,x) + u(t,x)\nabla u(t,x) = -\nabla p(t,x) & \text{balance of momentum,} \\ \operatorname{div}\, u(t,x) = 0 & \text{incompressibility condition,} \\ u(0,x) = \bar{u}(x) & \text{initial condition,} \end{cases}$$

where t is the time, $x \in \mathbb{R}^N$ is the *Eulerian* space variable, $u = (u_1,\dots,u_N)$ is the fluid velocity, p is the scalar pressure, it is possible to associate, under some regularity assumptions, a Lagrangian flow

$$\mathtt{x} : [0,\infty)\times\mathbb{R}^N \to \mathbb{R}^N, \qquad (t,x) \mapsto \mathtt{x}(t,x),$$

which describes the trajectory of the particle which is initially located at point x. The function $\mathtt{x}(t,x)$ is determined by solving the Cauchy problem:

$$\begin{cases} \frac{\partial \mathtt{x}}{\partial t}(t,x) = u(t,\mathtt{x}(t,x)), \\ \mathtt{x}(0,x) = x. \end{cases} \tag{31}$$

The notion of Lagrangian representation introduced in Definition 3 is very close to the what happens in the transport equation and in the Euler system. Indeed, we have a (scalar) field $\lambda(t,x)$ (which plays the same role as the vector field b in the transport equation or the velocity field u in the Euler system) and a set $\mathcal{W}$ of particles (the *waves*) such that

a) each wave moves on a trajectory which satisfies the ODE (26), exactly as each particle satisfies the ODE (30) in the transport equation and (31) in the Euler system;
b) the (distributional) derivative $v(t) := D_x u(t,\cdot)$ of u at any fixed time t satisfies the transport PDE
$$v_t + (\lambda(t,x)v)_x = 0; \tag{32}$$
c) the flux $\mathtt{x}$ transports the distributional derivative v of u along characteristic curves.

Up to now, as far as we know, the notion of Lagrangian representation has never been singled out as a tool for the analysis of general hyperbolic systems of conservation laws, even if the weak derivative $v = D_x u$ satisfies a transport equation similar to (32). Indeed, the main difficulty consists in the fact that, both in the transport equation and in the Euler one, the vector field which generate the flux $\mathtt{x}$ (respectively $b(t,x)$ and $u(t,x)$) shares some *incompressibility* (or nearly incompressibility) property, which are almost necessary to have the uniqueness of the solutions. On the contrary, for a system of conservation laws, the (scalar) field $\lambda(t,x)$ is, in general, non-incompressible, as appears from Example 2, thus yielding non-uniqueness issues. The proof of the existence of at least one Lagrangian representation presented in Chapter 5 of this thesis shows that we are able to select, among many, a "correct" flow $\mathtt{x}$, in the sense that it satisfies (the vector version of) Points a), b), c) above.

We conclude this brief overview about the Lagrangian point of view in conservation laws, explaining why we think that the existence of a Lagrangian representation for the Cauchy problem (1), (2) is important. First of all it is interesting from a theoretical point of view, because it provides the *continuous* counterpart in the exact solution to (1), (2) to the well established theory of wavefront approximations; indeed the family of trajectories $\{\mathtt{x}(t,w),\ w \in \mathcal{W}\}$ can be regarded as an (continuous) family of infinitesimal wavefronts, traveling with the (generalized) characteristic speeds.

Besides this purely theoretical aspect, we think that our Lagrangian approach to conservation laws can lead to a deeper understanding of the behavior of the solutions in the general setting, when the characteristic fields are not GNL/LD. In particular the Lagrangian description of the solution will allow to prove regularity results, similar to those already known for the GNL/LD case. More precisely, it is well known that the solution $u(t,x)$ is a BV function of the two variables (t,x), also in the general setting. Hence is shares the regularity properties of general BV functions. In particular either u is approximately continuous or it has an approximate jump at each point (t,x), with the exception of a set $\mathcal{N}$ whose one-dimensional Hausdorff measure is zero. In the genuinely non linear case it is proved (see, for instance, [**BL99**] and [**BY**]) that the set $\mathcal{N}$ is countable and u is continuous (not just approximately continuous) outside $\mathcal{N}$ and outside a countable family Γ of Lipschitz shock curves; moreover at any point $\Gamma \setminus \mathcal{N}$, the solution has left and right limits (not just approximate limits). It is thus natural to expect that the same kind of regularity holds also in the non-convex setting and, indeed, this can be obtained through the tools provided by the Lagrangian description of the solution.

Another immediate application of the Lagrangian formulation for conservation laws is that it will allow to prove all the interesting interaction estimates and to define all the Glimm-type functionals, looking directly at the exact solution $u(t,\cdot) = S_t\bar{u}$ (see Theorem 1), avoiding the analysis of the approximate (Glimm or wavefront) solutions, as it has been usually done up to now.

However, due to time constraints, we do not insert in this thesis such kind of results which will appear in a forthcoming paper [**BM15a**].

As a final remark, we observe that the only missing point to complete the cornerstones of the theory of strictly hyperbolic conservation laws in the general setting is a purely hyperbolic proof of the stability of the solutions (already provided, through parabolic methods, in the fundamental paper [**BB05**]), in the same spirit as [**BCP00**], [**LY99**], [**BLY99**].

Structure of the thesis

The thesis is organized as follows.

In Chapter 1 we collect some notations and mathematical preliminaries which will be used throughout the thesis. In particular we present results about the convex/concave envelope of a function $f : \mathbb{R} \to \mathbb{R}$ and we will recall some well-known results in measure theory, in the theory of BV functions of one variable and in the theory of monotone multi-functions.

Chapter 2 is still devoted to introduce some preliminary results, this time about the theory of Hyperbolic Conservation Laws. We will focus on those results which will be used in the subsequent chapters, in particular on the construction of the vanishing viscosity solution to the Riemann problem, the construction of the Glimm approximate solutions and the definitions of the Lyapunov functionals already present in the literature.

In Chapter 3 we present and prove the first result of this thesis, namely Theorem A. As we have already pointed out, Theorem A is the final outcome of papers [**BM14a**], [**BM14b**], [**BM15b**]. In particular we will present the result in the most general setting, namely the one considered in [**BM15b**], with a sharper definition of the potential $\mathfrak{Q}$ which is needed in the proof of the convergence of the Glimm scheme.

Aim of Chapter 4 is the proof of Theorem B, namely the estimate on the rate of convergence of the Glimm scheme. The result of this chapter can be found in the paper [**MB15**].

Finally Chapter 5 contains the third and last result of this thesis, namely Theorem C. As we have already pointed out, this Chapter is, in some sense, a *work in progress*. The proof of Theorem C presented in Chapter 5 is complete. However, due to time constraints, we decided not to insert some further results and corollaries which could be obtained with little effort from C. An extensive discussion of the matter will appear in [BM15a].

CHAPTER 1

Mathematical preliminaries

In this chapter we collect some notations and mathematical preliminaries which will be used throughout the thesis. In particular, in Section 1.1 we fix some notations we will widely use in the following. In Section 1.2 we present some results about the convex/concave envelope of a function $f:\mathbb{R}\to\mathbb{R}$. In Section 1.3 some tools from Measure Theory are introduces, while in Section 1.4 a brief overview on BV functions in one variable is given. Finally in Section 1.5 the definition of monotone multi-functions together with some of their properties can be found.

1.1. Notations

We fix here, for the usefulness of the reader, some notations which will be used throughout the thesis.

- The restriction of a map f to a subset A of its domain is denoted by $f|_A$.
- Given a totally ordered set $(A,\preceq)$, we define a partial pre-ordering on 2^A setting, for any $I,J\subseteq A$,

$$I\prec J \text{ if and only if for any } a\in I, b\in J \text{ it holds } a\prec b.$$

  We will also write $I\preceq J$ if either $I\prec J$ or $I=J$, i.e. we add the diagonal to the relation, making it a partial ordering.
- Given two sets $A\subseteq B$, the characteristic function of A on B is denoted by

$$\chi_A : B\to\{0,1\}, \qquad \chi_A(x)=\begin{cases}1 & \text{if } x\in A,\\ 0 & \text{if } x\in B\setminus A.\end{cases}$$

- For any $s\in\mathbb{R}$, define

$$\mathbf{I}(s):=\begin{cases}(0,s] & \text{if } s\geq 0,\\ [s,0) & \text{if } s<0.\end{cases} \tag{1.1}$$

- Let X be any (nonempty) set and let $f:\mathbf{I}(s')\to X$, $g: s'+\mathbf{I}(s'')\to X$;
  - if $s's''\geq 0$ and $f(s')=g(s')$, define

$$f\cup g:\mathbf{I}(s'+s'')\to X, \qquad \big(f\cup g\big)(x):=\begin{cases}f(x) & \text{if } x\in\mathbf{I}(s'),\\ g(x) & \text{if } x\in s'+\mathbf{I}(s'');\end{cases} \tag{1.2}$$

  - if $s's''<0$, define

$$f\vartriangle g:\mathbf{I}(s'+s'')\to X, \qquad (f\vartriangle g)(x):=\begin{cases}f(x) & \text{if } |s'|\geq|s''|,\ x\in\mathbf{I}(s'+s''),\\ g(x) & \text{if } |s'|<|s''|,\ x\in\mathbf{I}(s'+s'').\end{cases} \tag{1.3}$$

- Given an interval $I\subseteq\mathbb{R}$, a *piecewise constant (resp. affine) map* is a map $f:I\to\mathbb{R}$ such that I can be written as a finite union of intervals $I=I_1\cup\dots\cup I_n$ and f is constant (resp. affine) on each I_j, $j=1,\dots,n$.

- Given a C^1 map $g : \mathbb{R} \to \mathbb{R}$ and an interval $I \subseteq \mathbb{R}$, possibly made by a single point, let us define the Rankine-Hugoniot speed
$$\sigma^{\mathrm{rh}}(g, I) := \begin{cases} \dfrac{g(\sup I) - g(\inf I)}{\sup I - \inf I}, & \text{if } I \text{ is not a singleton,} \\ \dfrac{dg}{du}(I), & \text{if } I \text{ is a singleton.} \end{cases}$$
- The Lebesgue measure on (a subset of) $\mathbb{R}^d$ is denoted by $\mathcal{L}^d$.
- If $(X, \mathcal{A})$ is a measure space (see Section 1.3), the Dirac's delta (viewed as a measure) in a point $x \in X$ is denoted by δ_x.
- If μ is a measure on $(X, \mathcal{A})$, the integral of a measurable function f on X w.r.t. μ is denoted by $\int_X f(x)\mu(dx)$.
- The L^∞ norm of a map $g : [a, b] \to \mathbb{R}^n$ will be denoted either by $\|g\|_\infty$ or by $\|g\|_{L^\infty([a,b])}$, if we want to stress the domain of g; similar notation for the L^1-norm.

1.2. Convex envelopes and secant lines

In this section collect some results about convex envelopes of continuous functions and slopes of secant lines; these results are frequently used in the thesis.

We recall here the notion of *convex envelope* of a continuous function $g : \mathbb{R} \to \mathbb{R}$ and we state some results about convex envelops.

Definition 1.1. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and $[a, b] \subseteq \mathbb{R}$. We define *the convex envelope of g in the interval $[a, b]$* as

$$\operatorname*{conv}_{[a,b]} g(u) := \sup \left\{ h(u) \;\middle|\; h : [a, b] \to \mathbb{R} \text{ is convex and } h \leq g \right\}.$$

A similar definition holds for *the concave envelope of g in the interval $[a, b]$* denoted by $\operatorname*{conc}_{[a,b]} g$. All the results we present here for the convex envelope of a continuous function g hold, with the necessary changes, for its concave envelope.

Adopting the language of Hyperbolic Conservation Laws, we give the next definition.

Definition 1.2. Let g be a continuous function on $\mathbb{R}$, let $[a, b] \subseteq \mathbb{R}$ and consider $\operatorname{conv}_{[a,b]} g$. A *shock interval* of $\operatorname{conv}_{[a,b]} g$ is an open interval $I \subseteq [a, b]$ such that for each $u \in I$, $\operatorname{conv}_{[a,b]} g(u) < g(u)$.

A *maximal shock interval* is a shock interval which is maximal with respect to set inclusion.

A *shock point* is any $u \in [a, b]$ belonging to a shock interval. A *rarefaction point* is any point $u \in [a, b]$ which is not a shock point, i.e. any point such that $\operatorname{conv}_{[a,b]} g(u) = g(u)$.

The following theorem is classical and provides a description of the regularity of the convex envelope of a given function g. For a self contained proof (of a bit less general result), see Theorem 2.5 of [BM14a].

Theorem 1.3. *Let $g : [a, b] \to \mathbb{R}$ be a Lipschitz function.*

(1) The convex envelope $\operatorname*{conv}_{[a,b]} g$ of g in the interval $[a, b]$ is Lipschitz on $[a, b]$ and

$$\operatorname{Lip}\Big(\operatorname*{conv}_{[a,b]} g\Big) \leq \operatorname{Lip}(g);$$

if g is differentiable at $u \in (a, b)$, then $\operatorname{conv}_{[a,b]} g$ is differentiable at u and

$$\frac{dg}{du}(\bar{u}) = \frac{d \operatorname{conv}_{[a,b]} g}{du}(\bar{u});$$

moreover, $u \mapsto d\,\mathrm{conv}_{[a,b]}\, g(u)/du$ is defined a.e. and it is a monotone function in the sense of Definition 1.40.

(2) *If $\frac{dg}{du}$ is a BV function, then $\frac{d}{du}\mathrm{conv}_{[a,b]}\, g$ is a BV function and for any $a \le u_1 < u_2 \le b$,*

$$\text{e.Tot.Var.}\left(\frac{d\,\mathrm{conv}_{[a,b]}\, g}{du}; (u_1,u_2)\right) \le \text{e.Tot.Var.}\left(\frac{dg}{du}; (u_1,u_2)\right).$$

(3) *If $g \in C^1([a,b])$, then $\operatorname*{conv}_{[a,b]} g \in C^1([a,b])$ and, for any point $u \in (a,b)$ such that $g(u) = \operatorname*{conv}_{[a,b]} g(u)$, it holds*

$$\frac{d}{du} g(u) = \frac{d}{du}\operatorname*{conv}_{[a,b]} g(u).$$

(4) *If $g \in C^{1,1}([a,b])$, then $\operatorname*{conv}_{[a.b]} g \in C^{1,1}([a,b])$ and*

$$\mathrm{Lip}\left(\frac{d}{du}\operatorname*{conv}_{[a,b]} g\right) \le \mathrm{Lip}\left(\frac{dg}{du}\right).$$

By "$C^1([a,b])$" we mean that $\operatorname*{conv}_{[a,b]} g$ is C^1 on (a,b) in the classical sense and that in a (resp. b) the right (resp. the left) derivative exists.

We now state some useful results about convex envelopes, which we frequently use in the paper.

Proposition 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $a < \bar{u} < b$. If $\mathrm{conv}_{[a,b]} f(\bar{u}) = f(\bar{u})$, then*

$$\operatorname*{conv}_{[a,b]} f = \operatorname*{conv}_{[a,\bar{u}]} f \cup \operatorname*{conv}_{[\bar{u},b]} f.$$

Proof. See Proposition 2.7 of [BM14a]. □

Corollary 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $a < \bar{u} < b$. Assume that $\bar{u}$ belongs to a maximal shock interval (u_1, u_2) with respect to $\mathrm{conv}_{[a,b]} f$. Then $\mathrm{conv}_{[a,\bar{u}]} f|_{[a,u_1]} = \mathrm{conv}_{[a,b]} f|_{[a,u_1]}$.*

Proposition 1.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous; let $a < \bar{u} < b$. Then*

(1) $\left(\frac{d}{du}\mathrm{conv}_{[a,\bar{u}]} f\right)(u+) \ge \left(\frac{d}{du}\mathrm{conv}_{[a,b]} f\right)(u+)$ *for each $u \in [a,\bar{u})$;*
(2) $\left(\frac{d}{du}\mathrm{conv}_{[a,\bar{u}]} f\right)(u-) \ge \left(\frac{d}{du}\mathrm{conv}_{[a,b]} f\right)(u-)$ *for each $u \in (a,\bar{u}]$;*
(3) $\left(\frac{d}{du}\mathrm{conv}_{[\bar{u},b]} f\right)(u+) \le \left(\frac{d}{du}\mathrm{conv}_{[a,b]} f\right)(u+)$ *for each $u \in [\bar{u},b)$;*
(4) $\left(\frac{d}{du}\mathrm{conv}_{[\bar{u},b]} f\right)(u-) \le \left(\frac{d}{du}\mathrm{conv}_{[a,b]} f\right)(u-)$ *for each $u \in (\bar{u},b]$.*

The above statement is identical to Proposition 2.9 of [BM14a], to which we refer for the proof.

Proposition 1.7. *Let $g : \mathbb{R} \to \mathbb{R}$ be C^1 and $a < \bar{u} < b$. Then*

(1) *for each $u_1, u_2 \in [a,\bar{u}]$, $u_1 < u_2$,*

$$\left(\frac{d}{du}\operatorname*{conv}_{[a,\bar{u}]} g\right)(u_2) - \left(\frac{d}{du}\operatorname*{conv}_{[a,\bar{u}]} g\right)(u_1) \ge \left(\frac{d}{du}\operatorname*{conv}_{[a,b]} g\right)(u_2) - \left(\frac{d}{du}\operatorname*{conv}_{[a,b]} g\right)(u_1);$$

(2) *for each $u_1, u_2 \in [\bar{u},b]$, $u_1 < u_2$,*

$$\left(\frac{d}{du}\operatorname*{conv}_{[\bar{u},b]} g\right)(u_2) - \left(\frac{d}{du}\operatorname*{conv}_{[\bar{u},b]} g\right)(u_1) \ge \left(\frac{d}{du}\operatorname*{conv}_{[a,b]} g\right)(u_2) - \left(\frac{d}{du}\operatorname*{conv}_{[a,b]} g\right)(u_1),$$

where the derivative in the endpoints of the intervals are in the sense of right/left derivative.

Proof. See Proposition 2.10 of [BM14a]. $\square$

Proposition 1.8. *Let $a^0 \le a^1 \le \dots a^P$ be P real numbers. Let $f : [a^0, a^P] \to \mathbb{R}$ be a $C^{1,1}$ map. Set*

$$\sigma(\tau) := D \operatorname*{conv}_{[a^0,a^P]} f(\tau), \qquad \sigma^p(\tau) := D \operatorname*{conv}_{[a^{p-1},a^p]} f(\tau), \text{ for } p = 1, \dots, P.$$

Then for any constant $\sigma^* \in \mathbb{R}$,

$$\int_{a^0}^{a^P} \big|\sigma(\tau) - \sigma^*\big| d\tau \le \sum_{p=1}^{P} \int_{a^{p-1}}^{a^p} \big|\sigma^p(\tau) - \sigma^*\big|.$$

Proof. Define

$$b := \begin{cases} \inf\big\{\tau \in [a^0, a^P] \;\big|\; \sigma(\tau) \ge b\big\} & \text{if } \big\{\tau \in [a^0, a^P] \;\big|\; \sigma(\tau) \ge b\big\} \ne \emptyset, \\ a^P & \text{otherwise.} \end{cases}$$

It is easy to see that $\operatorname{conv}_{[a^0,a^P]} f(b) = f(b)$ and thus, by Proposition 1.4, if $b \in [a^{\bar p-1}, a^{\bar p}]$, then $\operatorname{conv}_{[a^{\bar p-1},a^{\bar p}]} f(b) = f(b)$. Therefore

$$\begin{aligned}
\int_{a^0}^{b} \sigma(\tau) d\tau &= \operatorname*{conv}_{[a^0,a^P]} f(b) - \operatorname*{conv}_{[a^0,a^P]} f(0) \\
&= f(b) - f(0) \\
&= f(b) - f(a^{\bar p-1}) + \sum_{p=1}^{\bar p-1} f(a^p) - f(a^{p-1}) \\
&= \Big(\operatorname*{conv}_{[a^{\bar p-1},a^{\bar p}]} f(b) - \operatorname*{conv}_{[a^{\bar p-1},a^{\bar p}]} f(a^{\bar p-1}) \Big) + \sum_{p=1}^{\bar p-1} \Big(\operatorname*{conv}_{[a^{p-1},a^p]} f(a^p) - \operatorname*{conv}_{[a^{p-1},a^p]} f(a^{p-1}) \Big) \\
&= \int_{a^{\bar p-1}}^{b} \sigma^p(\tau) d\tau + \sum_{p=1}^{\bar p-1} \int_{a^{p-1}}^{\min\{a^p,b\}} \sigma^p(\tau) d\tau
\end{aligned}$$

and thus

$$\begin{aligned}
\int_{a^0}^{b} |\sigma^* - \sigma(\tau)| d\tau &= \int_{a^0}^{b} \sigma^* - \sigma(\tau) d\tau \\
&= \int_{a^{\bar p-1}}^{b} \big(\sigma^* - \sigma^p(\tau)\big) d\tau + \sum_{p=1}^{\bar p-1} \int_{a^{p-1}}^{a^p} \big(\sigma^* - \sigma^p(\tau)\big) d\tau \qquad (1.4) \\
&\le \int_{a^{\bar p-1}}^{b} \big|\sigma^* - \sigma^{\bar p}(\tau)\big| d\tau + \sum_{p=1}^{\bar p-1} \int_{a^{p-1}}^{a^p} \big|\sigma^* - \sigma^p(\tau)\big| d\tau.
\end{aligned}$$

Similarly

$$\int_{b}^{a^P} \big|\sigma(\tau) - \sigma^*\big| d\tau \le \int_{b}^{a^{\bar p}} \big|\sigma^{\bar p}(\tau) - \sigma^*\big| + \sum_{p=\bar p+1}^{P} \int \big|\sigma^p(\tau) - \sigma^*\big| d\tau. \qquad (1.5)$$

The conclusion follows easily from (1.4) and (1.5). $\square$

PROPOSITION 1.9. *Let $g : [a, b] \to \mathbb{R}$ be a continuous map. Let $a \le c \le b$. Then*
$$\int_a^c \left| D \operatorname*{conv}_{[a,b]} g(\tau) - D \operatorname*{conv}_{[a,c]} g(\tau) \right| d\tau = g(c) - \operatorname*{conv}_{[a,b]} g(c).$$

PROOF. We already know that $D \operatorname{conv}_{[a,b]} g(\tau) \le D \operatorname{conv}_{[a,c]} g(\tau)$. Therefore
$$\begin{aligned}\int_a^c \left| D \operatorname*{conv}_{[a,b]} g(\tau) - D \operatorname*{conv}_{[a,c]} g(\tau) \right| d\tau &= \int_a^c \Big(D \operatorname*{conv}_{[a,c]} g(\tau) - D \operatorname*{conv}_{[a,b]} g(\tau) \Big) d\tau \\ &= g(c) - D \operatorname*{conv}_{[a,b]} g(c).\end{aligned}$$
□

COROLLARY 1.10. *If $g : [a, b] \to \mathbb{R}$ is Lipschitz and $a \le c \le b$, then*
$$\int_a^c \left| D \operatorname*{conv}_{[a,b]} g(\tau) - D \operatorname*{conv}_{[a,c]} g(\tau) \right| d\tau \le \operatorname{Lip}(g) |b - c|.$$

PROOF. The proof is an easy consequence of previous proposition and Theorem 1.3. □

PROPOSITION 1.11. *Let $g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be Lipschitz functions. Let $a, b \in \mathbb{R}$, $a < b$. Then it holds*
$$\begin{aligned}&\Big\| \operatorname*{conv}_{[a,b]} g - \operatorname*{conv}_{[a,b]} h \Big\|_\infty \le \big\| g - h \big\|_\infty, \\ &\left\| \frac{d}{du} \operatorname*{conv}_{[a,b]} g - \frac{d}{du} \operatorname*{conv}_{[a,b]} h \right\|_\infty \le \left\| \frac{dg}{du} - \frac{dh}{du} \right\|_\infty, \\ &\left\| \frac{d \operatorname{conv}_{[a,b]} g}{d\tau} - \frac{d \operatorname{conv}_{[a,b]} h}{d\tau} \right\|_1 \le \left\| \frac{dg}{d\tau} - \frac{dh}{d\tau} \right\|_1.\end{aligned}$$

PROOF. See Proposition 2.12 of [BM14b] and Lemma 3.1 of [Bia03]. □

PROPOSITION 1.12. *Let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz map. Let $[a, b] \subseteq \mathbb{R}$ and assume that I is an maximal open shock interval for $\operatorname{conv}_{[a,b]} g$. Set*
$$\sigma := \frac{d \operatorname{conv}_{[a,b]} g}{du}(I).$$
Then for any $\varepsilon > 0$,
$$\mathcal{L}^1 \Big(\Big\{ u \in \big(\inf I, \inf I + \varepsilon \big) \ \Big| \ g'(u) \ge \sigma \Big\} \Big) > 0$$
and
$$\mathcal{L}^1 \Big(\Big\{ u \in \big(\sup I - \varepsilon, \sup I \big) \ \Big| \ g'(u) \le \sigma \Big\} \Big) > 0.$$

PROOF. Let us prove only the first inequality, the second one being completely similar. Assume by contradiction that there is $\varepsilon > 0$ such that
$$\mathcal{L}^1 \Big(\Big\{ u \in \big(\inf I, \inf I + \varepsilon \big) \ \Big| \ g'(u) \ge \sigma \Big\} \Big) = 0.$$
It holds
$$\int_{\inf I}^{\inf I + \varepsilon} \sigma dz = \operatorname*{conv}_{[a,b]} g(\inf I + \varepsilon) - \operatorname*{conv}_{[a,b]} g(\inf I) \le g(\inf I + \varepsilon) - g(\inf I) = \int_{\inf I}^{\inf I + \varepsilon} g'(z) dz.$$
Hence
$$\int_{\inf I}^{\inf I + \varepsilon} \big[g'(z) - \sigma \big] dz \ge 0,$$
a contradiction. □

PROPOSITION 1.13. *Let $f : [a,b] \to \mathbb{R}$ be a Lipschitz map. Let $\{I_n\}_n$ be a countable family of non-trivial intervals, $I_n \subseteq [a,b]$, $\mathcal{L}^1(I_n) > 0$. Define*

$$g(x) := \begin{cases} \mathrm{conv}_{I_n} f(x) & \text{if } x \in I_n, \\ f(x) & \text{otherwise.} \end{cases}$$

Then g is Lipschitz and for any n, $g(\inf I_n) = f(\inf I_n)$, $g(\sup I_n) = f(\sup I_n)$.

The proof is not difficult and thus it is omitted.

We conclude this section with two results related to the slope of the secant line of a function g between two given points $a \leq b$. Their proofs are very easy and thus they are omited. Using the language of Hyperbolic Conservation Laws, we will call this slope the *Rankine-Hugoniot speed given by the map g to the interval $[a,b]$.*

PROPOSITION 1.14. *Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^{1,1}$ function and let $a \in \mathbb{R}$. Then the map*

$$x \mapsto \begin{cases} \dfrac{g(x) - g(a)}{x - a}, & \text{if } x \neq 0, \\ g'(a), & \text{if } x = 0 \end{cases}$$

is Lipschitz on $\mathbb{R}$, with Lipschitz constant equal to $\mathrm{Lip}(g')$.

PROPOSITION 1.15. *Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^{1,1}$ function, let $[a,b] \subseteq \mathbb{R}$, $\bar{u} \in [a,b]$ such that $\underset{[a,b]}{\mathrm{conv}}\, g(\bar{u}) = g(\bar{u})$. Then for any $u \in [a,b]$,*

- *if $u \in [a,\bar{u}]$, then*
$$\sigma^{\mathrm{rh}}(g,[u,\bar{u}]) \leq \sigma^{\mathrm{rh}}(g,[u,b]);$$
- *if $u \in [\bar{u},b]$, then*
$$\sigma^{\mathrm{rh}}(g,[\bar{u},u]) \geq \sigma^{\mathrm{rh}}(g,[a,u]).$$

1.3. Some tools from Measure Theory

In this section we collect some notations and results in measure theory which will be used in the next chapters. Our main reference is [AFP00].

1.3.1. Abstract measure theory. Let X be a set and let $\mathcal{A}$ be a σ-algebra on X. We call $(X,\mathcal{A})$ a *measure space*. A *positive measure* μ on $(X,\mathcal{A})$ is a function $\mu : \mathcal{A} \to [0,\infty]$ such that $\mu(\emptyset) = 0$ and μ is σ-additive, i.e. for any sequence $\{E_n\}_n \subseteq \mathcal{A}$, $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$. If μ is a positive measure on $(X,\mathcal{A})$, we will say that a set $E \subseteq X$ is *μ-negligible* if there exists $Z \in \mathcal{A}$ such that $E \subseteq Z$ and $\mu(Z) = 0$. We say that a property $P(x)$ depending on the point $x \in X$ holds *μ-almost everywhere* in X if the set where P fails is a μ-negligible set. If $\mu = \mathcal{L}^d$ is the Lebesgue measure on (a subset of) $\mathbb{R}^d$, we will simply say the the property *$P(x)$ holds almost everywhere.*

A *real-valued measure* (or simply a *measure*) μ on $(X,\mathcal{A})$ is a function $\mu : \mathcal{A} \to \mathbb{R}$ such that $\mu(\emptyset) = 0$ and μ is σ-additive, i.e. for any sequence $\{E_n\}_n \subseteq \mathcal{A}$, the series $\sum_n \mu(E_n)$ is absolutely convergent and $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$. If μ is a measure on X and $f : X \to [-\infty,+\infty]$ is a μ-measurable map, we denote the integral of f over X w.r.t. the measure μ with $\int_X f(x)\mu(dx)$. If $\mu = \mathcal{L}^d$ is the Lebesgue measure on (a subset of) $\mathbb{R}^d$ we simply write $\int_{\mathbb{R}^d} f(x)dx$. The measure ν on $(X,\mathcal{A})$ defined by $\nu(E) := \int_E \mu(dx)$ is denoted by $f\mu$. The *total variation* of a measure μ on $(X,\mathcal{A})$ is the positive measure on $(X,\mathcal{A})$ defined by

$$|\mu|(E) := \sup\left\{ \sum_{h=1}^{\infty} |\mu(E_h)| \;\middle|\; E_h \in \mathcal{A} \text{ pairwise disjoint, } E = \bigcup_{h=0}^{\infty} E_h \right\}.$$

The set $\mathcal{M}(X)$ of all measure on $(X, \mathcal{A})$ with the norm $\|\mu\| := |\mu|(X)$ turns out to be a Banach space.

DEFINITION 1.16. Let $(X, \mathcal{A})$ be a measure space. Let μ be a measure on $(X, \mathcal{A})$ and let ν be a positive measure on $(X, \mathcal{A})$. We say that μ is *absolutely continuous w.r.t.* ν and we write $\mu \ll \nu$ if for every $B \in \mathcal{A}$ the following implication holds:

$$\nu(B) = 0 \quad \Longrightarrow \quad |\mu|(B) = 0.$$

DEFINITION 1.17. If μ, ν are positive measure on $(X, \mathcal{A})$, we say that they are *mutually singular*, and write $\mu \perp \nu$ if there exists $E \in \mathcal{E}$ such that

$$\mu(E) = 0, \qquad \nu(X \setminus E) = 0.$$

If μ, ν are measure on $(X, \mathcal{A})$ we say that they are *mutually singular* if $|\mu|$, $|\nu|$ are.

THEOREM 1.18 (Radon-Nikodim). *Let μ be positive measure on the measure space $(X, \mathcal{A})$. Let ν be a measure on $(X, \mathcal{A})$. If μ is σ-finite, then there is a unique pair of measures ν^a, ν^s such that*

$$\nu^a \ll \mu, \qquad \nu^s \perp \mu, \qquad \nu = \nu^a + \nu^s.$$

Moreover, there is a unique function $f \in L^1(X, \mathcal{A}, \mu)$ such that

$$\nu^a = f\mu.$$

The function f (defined up to μ-negligible sets) is called the density of ν w.r.t. μ *and it is denoted by $\frac{d\nu}{d\mu}$.*

DEFINITION 1.19. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measure spaces. Let $f : X \to Y$ be any map such that $f^{-1}(B) \in \mathcal{A}$ for any $B \in \mathcal{B}$. If μ is a measure on $(X, \mathcal{A})$, we define the *push-forward* $f_\sharp\mu$ of μ through f as the measure on $(Y, \mathcal{B})$ defined by

$$f_\sharp\mu(B) := \mu\big(f^{-1}(B)\big).$$

It is well known that for any $\phi \in L^1(Y, \mathcal{B}, f_\sharp\mu)$ it holds

$$\int_Y \phi(y) f_\sharp\mu(dy) = \int_X \phi\big(f(x)\big)\mu(dx).$$

The following lemma describe the relation between absolute continuity and the push-forward of measures.

LEMMA 1.20. *Let μ be a measure on $(X, \mathcal{A})$ and let ν be a positive measure on $(X, \mathcal{A})$. Let $(Y, \mathcal{B})$ be a measure space and let $f : X \to Y$ be a map such that $f^{-1}(B) \in \mathcal{A}$ for any $B \in \mathcal{B}$. If $\mu \ll \nu$, then $f_\sharp\mu \ll f_\sharp\nu$.*

PROOF. The proof is immediate from the previous definitions. $\square$

We conclude this section with the following two trivial lemmas, which will be used in Chapter 5.

LEMMA 1.21. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $f_n, f : X \to \mathbb{R}$, $n \in \mathbb{N}$. Assume that $f_n(x) \to f(x)$ for μ-a.e. $x \in X$. Let $J \subseteq \mathbb{R}$ be an interval such that*

$$\mu\left(\left\{f^{-1}(\inf J), f^{-1}(\sup J)\right\}\right) = 0.$$

Then $\chi_{f_n^{-1}(J)} \to \chi_{f^{-1}(J)}$ μ-a.e. and in L^1.

Proof. Set

$$E := \{x \in X \mid f_n(x) \nrightarrow f(x),\ f(x) \neq \inf J,\ f(x) \neq \sup J\}.$$

By the hypothesis, $\mu(E) = 0$. If $x \notin E$, $f_n(x) \to f(x)$. If $x \in (\inf J, \sup J)$, then, for $n \gg 1$, $f_n(x) \in (\inf J, \sup J)$ and thus $\chi_{f_n^{-1}(J)}(x) \to \chi_{f^{-1}(J)}(x)$. Similarly, if $x \notin [\inf J, \sup J]$, then for $n \gg 1$, $f_n(x) \notin [\inf J, \sup J]$ and thus $\chi_{f_n^{-1}(J)}(x) \to \chi_{f^{-1}(J)}(x)$. By the Dominated Convergence Theorem we get L^1 convergence. □

Lemma 1.22. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $f_n, f : X \to \mathbb{R}$ be a sequence of measurable functions such that $f_n(x) \to f(x)$ for μ-a.e. $x \in X$ (here f_n is defined for all $x \in X$, not just for μ-a.e. $x \in X$). Let $\bar{x} \in X$ be a fixed point where $f_n(\bar{x}) \to f(\bar{x})$. If*

$$f^{-1}\big(f(\bar{x})\big) = \{\bar{x}\},$$

then

$$\mathcal{L}^1\Big(f_n^{-1}\big(f_n(\bar{x})\big)\Big) \to 0 \text{ as } n \to \infty.$$

Proof. Set $J_\delta := [f(\bar{x}) - \delta, f(\bar{x}) + \delta]$. Clearly $\mathcal{L}^1\big(f^{-1}(J_\delta)\big) \to 0$ as $\delta \to 0$ and for a.e. $\delta > 0$,

$$\mathcal{L}^1\left(\Big\{f^{-1}\big(f(\bar{x}) - \delta\big),\ f^{-1}\big(f(\bar{x}) + \delta\big)\Big\}\right) = 0.$$

Therefore, by previous lemma,

$$\mathcal{L}^1\Big(f_n^{-1}(J_\delta)\Big) \to \mathcal{L}^1\Big(f^{-1}(J_\delta)\Big) \text{ as } n \to \infty.$$

Since $f_n(\bar{x}) \to f(\bar{x})$ as $n \to \infty$, for any $\delta > 0$ and for any $n \gg 1$ (depending on δ), $f_n(x) \in J_\delta$ and thus $f_n^{-1}\big(f_n(\bar{x})\big) \subseteq f_n^{-1}(J_\delta)$. Hence

$$\limsup_{n\to\infty} \mathcal{L}^1\Big(f_n^{-1}\big(f_n(\bar{x})\big)\Big) \leq \mathcal{L}^1\Big(f^{-1}(J_\delta)\Big).$$

Letting $\delta \to 0$, we get the conclusion. □

1.3.2. Measure on metric spaces. In this and the two next sections X is a locally compact separable (l.c.s.) metric space equipped with the σ-algebra $\mathcal{B}(X)$ (or simply $\mathcal{B}$) of its Borel subsets.

Definition 1.23. Let X be a l.c.s. metric space, $\mathcal{B}(X)$ its Borel σ-algebra.

(a) A positive measure on $(X, \mathcal{B}(X))$ is called a *Borel measure*. If a Borel measure is finite on compact sets, it is called a *positive Radon measure*.
(b) A real valued set function on the relatively compact Borel subsets of X that is a measure on $(K, \mathcal{B}(K))$ for any compact set $K \subseteq X$ is called a *Radon measure* on X. Moreover, if $\mu : \mathcal{B}(X) \to \mathbb{R}$ is a measure, we say that it is a *finite Radon measure*.

Given a l.c.s. metric space we introduce the following two spaces. The space $C_c(X)$ is the set of all continuous function $X \to \mathbb{R}$ with compact support, while the space $C_0(X)$ is the completion of $C_c(X)$ w.r.t. the sup -norm. If X is compact the two spaces coincide. If X is not compact, $C_c(X)$ is the locally convex space whose topology is obtained considering a sequence of compact $A_n \nearrow X$ and imposing that the inclusions $j_n : C_c(A_n) \to C_c(X)$ are continuous. The link between Radon measures and compactly supported continuous functions is given by the following theorem

THEOREM 1.24 (Riesz). *Let X be a l.c.s. metric space; suppose that the functional $L : C_0(X) \to \mathbb{R}$ is additive and bounded, i.e. satisfies the following conditions:*

$$L(u+v) = L(u) + L(v), \qquad \text{for all } u, v \in C_0(X)$$

and

$$\|L\| := \sup\left\{ L(u) \;\middle|\; u \in C_0(X),\ |u| \leq 1 \right\} < \infty.$$

Then there is a unique finite Radon measure μ on X such that

$$L(u) = \int_X u(x)\mu(dx), \quad \text{for all } u \in C_0(X).$$

We conclude this section with the following definition.

DEFINITION 1.25. Let μ be a positive measure on the l.c.s. metric space X. The closed set of all points $x \in X$ such that $\mu(U) > 0$ for every neighborhood U of x is called the *support* of μ, denoted by suppμ. If μ is a measure, the *support* of μ is the support of $|\mu|$.

In general for a measure on a measurable space $(X, \mathcal{A})$ we say that μ *is concentrated on* S if $S \in \mathcal{A}$ and $\mu(X \setminus S) = 0$.

1.3.3. Weak* convergence of measures. We introduce now the notion of weak* convergence of measures and we prove some related results.

DEFINITION 1.26. Let X be a l.c.s. metric space, equipped with the Borel σ-algebra. Let $\mu \in \mathcal{M}(X)$ and $\{\mu^n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}(X)$. We say that $(\mu^n)_n$ weakly* converges to μ if

$$\lim_{n\to\infty} \int_X u(x)\mu^n(dx) = \int_X u(x)\mu(dx)$$

for any $u \in C_0(X)$.

Notice that, by the Riesz's Theorem, we can identify the space $\mathcal{M}(X)$ as the dual space of $C_0(X)$. The notion of weak* convergence of measures coincides with the standard notion of weak* convergence in the dual of a Banach space. The following theorem is thus just an application of the general theory about weak* compactness criteria.

THEOREM 1.27 (Compactness criterion for finite Radon measure). *If $(\mu^n)_n$ is a sequence of finite Radon measures on the l.c.s. metric space X with $\sup\{|\mu^n| \mid n \in \mathbb{N}\} < \infty$, then it has a weakly* converging subsequence. Moreover the map $\mu \mapsto |\mu|(X)$ is lower semicontinuous w.r.t. the weak* convergence.*

We now prove two lemmas which describe the relation between weak* convergence and the notions of absolute continuity and push-forwards.

LEMMA 1.28. *Let (μ^n) be a sequence of finite Radon measure on the l.c.s. metric space X and assume that (μ^n) weakly* converges to a measure $\mu \in \mathcal{M}(X)$. Let ν be a σ-finite Borel measure on X. Assume that $\mu^n \ll \nu$ for any n and let f^n be the density function of μ^n w.r.t. ν, i.e. $\mu^n = f^n\nu$. If $|f^n| \leq C$ for some constant C independent of n then $\mu \ll \nu$.*

PROOF. Since $|f^n| \leq C$, there exists a function $f \in L^\infty(X)$ such that (f^n) weakly* converges to f in $L^\infty(X, \mathcal{B}, \nu)$ up to subsequences. Therefore, for any $\phi \in C_0(X)$, we have

$$\int_X \phi(x)\mu^n(dx) = \int_X \phi(x)f^n(x)\nu(x) \to \int_X \phi(x)f(x)\nu(x).$$

Since

$$\int_X \phi(x)\mu^n(dx) \to \int_X \phi(x)\mu(dx)$$

it must hold $\mu = f\nu$. Therefore for any $B \in \mathcal{B}(X)$ such that $\nu(B) = 0$ we have

$$0 = \lim_n \int_X \chi_B(x)\mu^n(dx) = \lim_n \int_X \chi_B(x) f^n(x)\nu(dx) = \int_X \chi_B(x) f(x)\nu(dx) = \int_X \chi_B(x)\mu(dx),$$

and thus $\mu \ll \nu$. $\square$

LEMMA 1.29. *Let μ^n, μ, $n \in \mathbb{N}$, be finite Radon measure on the l.c.s. metric space $(X, \mathcal{B}(X))$. Let $(Y, \mathcal{B}(Y))$ be another l.c.s. metric space. Let $f : X \to Y$ be a continuous map. If (μ^n) weakly* converges to μ and $\mathrm{supp}\mu^n \subseteq K \subseteq X$, where K is a compact set not depending on n, then $(f_\sharp\mu^n)$ weakly* converges to $f_\sharp\mu$.*

PROOF. Since all the measures μ^n have support contained in a fixed compact set, we can assume that $f \in C_c(\Omega)$. Therefore for any $\phi \in C_0(\Omega)$, $\phi \circ f \in C_0(\Omega)$. The conclusion now follows applying the definition of weak* convergence of finite Radon measures. $\square$

We conclude this section with the following proposition.

PROPOSITION 1.30. *Let (μ^n) be a sequence of Radon measure on the l.c.s. metric space X weakly* converging to μ. Then*

(a) If the measures μ^n are positive, then for every lower semicontinuous $u : X \to [0, \infty]$

$$\liminf_{n\to\infty} \int_X u(x)\mu^n(dx) \geq \int_X u(x)\mu(dx)$$

and for every upper semicontinuous function $v : X \to [0, \infty]$ with compact support

$$\limsup_{n\to\infty} \int_X v(x)\mu^n(dx) \leq \int_X v(x)\mu(dx).$$

(b) If $(|\mu^n|)$ weakly converges to λ, then $\lambda \geq |\mu|$. Moreover, if E is a relatively compact Borel set such that $\lambda(\partial E) = 0$, then $\mu^n(E) \to \mu(E)$ as $n \to \infty$. More generally,*

$$\int_X u(x)\mu(dx) = \lim_{n\to\infty} \int_X u(x)\mu^n(dx)$$

for any bounded Borel function $u : X \to \mathbb{R}$ with compact support such that the set of its discontinuity points is λ-negligible.

For a proof, see [**AFP00**, Proposition 1.62]. As a particular case of previous proposition we get that

$$\mu(K) \geq \limsup_{n\to\infty} \mu^n(K)$$

for any compact set $K \subseteq X$ and

$$\mu(A) \leq \limsup_{n\to\infty} \mu^n(A)$$

for any open $A \subseteq X$.

1.3.4. The disintegration theorem. In this section we present the well-known disintegration theorem. We start with the following definitions.

DEFINITION 1.31. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measure spaces. Let ν be a positive measure on Y. Let $y \mapsto \mu_y$ be a function which assigns to each $y \in Y$ a measure μ_y on X. We say that this map is *measurable* if $y \mapsto \mu_y(A)$ is measurable for any $A \in \mathcal{A}$.

If $y \mapsto \mu_y$ is a measurable measure-valued function, the *generalized product* of (μ_y) and ν is the measure μ on $X \times Y$ defined by

$$\mu(A) := \int_Y \left(\int_X \chi_A(x, y)\mu_y(dx) \right) \nu(dy), \qquad \text{for any } A \in \mathcal{A}.$$

We will write for simplicity

$$\mu = \int_Y \mu_y \nu(dy).$$

Notice that for every $f \in L^1(X \times Y, \mathcal{A}, \mu)$ it holds

$$\int_{X\times Y} f(x,y)\mu(dxdy) = \int_Y \left(\int_X f(x,y)\mu_y(dx) \right) \nu(dy).$$

THEOREM 1.32 (Disintegration Theorem). *Let X, Y be l.c.s. metric spaces. Let $\mu \in \mathcal{M}(X \times Y)$, $\nu \in \mathcal{M}(Y)$, $\nu \geq 0$. Let $f : X \times Y \to Y$ be the standard projection $f(x,y) = y$. If $f_\sharp|\mu| \ll \nu$, then there exists a family $\{\mu_y\}_{y\in Y}$ of measures on X such that*

$$\mu = \int_Y \mu_y \nu(dy).$$

Moreover, for ν-a.e. $y \in Y$, the measure μ_y is concentrated on $f^{-1}(y)$.

For a proof of the Disintegration Theorem see [**AFP00**, Theorem 2.28].

1.4. *BV* functions

In this section we summarize the definition of pointwise and essential total variation of a function of one real variable and we collect some results which will be useful later.

DEFINITION 1.33. Let $I \subseteq \mathbb{R}$ be a (possibly unbounded) interval in $\mathbb{R}$ (we do not specify if the extrema belong to I or not). For any function $u : I \mapsto \mathbb{R}$ the *pointwise variation* $\text{p.Tot.Var.}(u; I)$ of u in I is defined by

$$\text{p.Tot.Var.}(u; I) := \sup \left\{ \sum_{p=1}^{P-1} \big|u(x_{p+1}) - u(x_p)\big| \;\middle|\; P \in \mathbb{N},\ x_1, \dots, x_P \in I,\ x_1 \leq \dots \leq x_P \right\}.$$

Clearly, $\text{p.Tot.Var.}(u; I)$ is very sensitive to modifications of the values of u even at a single point. This suggest the following definition.

DEFINITION 1.34. Let $I \subseteq \mathbb{R}$ be an open (possibly unbounded) interval in $\mathbb{R}$. For any function $u : I \mapsto \mathbb{R}$ the *essential total variation* $\text{p.Tot.Var.}(u; I)$ of u in I is defined by

$$\text{e.Tot.Var.}(u; I) := \inf \Big\{ \text{p.Tot.Var.}(v; I) \;\Big|\; v = u \ \mathcal{L}^1 - \text{a.e. in } I \Big\}.$$

If $\text{e.Tot.Var.}(u; I) < \infty$ we will say that u *has bounded variation.* Any map v in the equivalence class of u such that $\text{e.Tot.Var.}(u; I) = \text{e.Tot.Var.}(v; I) = \text{p.Tot.Var.}(v; I)$ is called a *good representative (in the equivalence class) of u.*

If $I \subseteq \mathbb{R}$ is an interval (not necessarily open) and $u : I \to \mathbb{R}$ is a good representative in its equivalence class (for instance if u il left- or right-continuous), we will use the notation $\text{Tot.Var.}(u; I)$ to denote the essential total variation $\text{e.Tot.Var.}\big(u; \mathring{I}\big)$ of u in the interior of I.

The space of all bounded variation functions on I is denoted by $\text{BV}(I)$. It is well known (see for instance [**AFP00**, Chapter 3]) that if $u \in \text{BV}(I)$, then its distributional derivative Du is a finite Radon measure on I.

Notice that the pointwise total variation is defined for any interval $I \subseteq \mathbb{R}$, while the essential total variation is defined only for functions on an open interval.

LEMMA 1.35. *Let $I = (a,b) \subseteq \mathbb{R}$ be an open interval. Let $u \in BV(I)$. Then there exists a unique left-continuous map $u^l : I \to \mathbb{R}$ and a unique right-continuous map $u^r : I \to \mathbb{R}$ such that $u^l = u^r = u$ for a.e. $x \in I$. Moreover*

$$u^l(x) = c + Du\big((a,x)\big), \qquad u^r(x) = c + Du\big((a,x]\big),$$

where Du is the distributional derivative of u.

For a proof, see [AFP00, Theorem 3.28]. The map u^l (resp. u^r) is called the *left-continuous (resp. right-continuous) good representative of u.*

The following lemma and corollary are useful when one wants to estimate the total variation of a map u which is defined almost everywhere.

LEMMA 1.36. *Let $u \in L^1_{\mathrm{loc}}(I) \cap BV(I)$. Assume that it is a good representative in its class of equivalence, i.e.*

$$\text{p.Tot.Var.}(u; I) = \text{e.Tot.Var.}(u; I).$$

Let $E \subseteq I$, $\mathcal{L}^1(E) = 0$. Then it holds

$$\text{p.Tot.Var.}(u; I) = \sup\left\{ \sum_{p=1}^{P-1} \big|u(x_{p+1}) - u(x_p)\big| \;\middle|\; P \in \mathbb{N},\ x_1 < \cdots < x_P,\ x_p \in I \setminus E \right\}.$$

Sometimes we will write for simplicity

$$\text{p.Tot.Var.}(u; I \setminus E) := \sup\left\{ \sum_{p=1}^{P-1} \big|u(x_{p+1}) - u(x_p)\big| \;\middle|\; P \in \mathbb{N},\ x_1 < \cdots < x_P,\ x_p \in I \setminus E \right\}.$$

REMARK 1.37. If u is not a good representative, then the sup in the previous formula is in general different both from p.Tot.Var.$(u; I)$ (since E could contain some "crazy" point where u has a big variation) and from e.Tot.Var.$(u; I)$ (since $I \setminus E$ could contain some "crazy" point where u has a big variation).

PROOF. The inequality

$$\text{p.Tot.Var.}(u; I) \geq \sup\left\{ \sum_{p=1}^{P-1} \big|u(x_{p+1}) - u(x_p)\big| \;\middle|\; P \in \mathbb{N}\ x_p \in I \setminus E \text{ for any } p \text{ and } x_1 < \cdots < x_P \right\}$$

is clear since p.Tot.Var. is defined taking the sup on a biggest set. To prove the other inequality, let us argue as follows. Take any finite sequence $x_1 < \cdots < x_P$ in I. Now fix $\varepsilon > 0$ and construct another finite sequence

$$x'_1 \leq x_1 \leq x''_1 < x'_2 \leq x_2 \leq x''_2 < \cdots < x'_{P-1} \leq x_{P-1} \leq x''_{P-1} < x'_P \leq x_P \leq x''_P$$

as follows. For any $p = 1, \dots, P$, if $x_p \in I \setminus E$, then set $x'_p = x''_p = x_p$. If $x_p \in I \cap E$, then distinguish two cases.

(1) Assume first that x_p is a continuity point for u. In this case there is a point $x'_p = x''_p \in I \setminus E$ sufficiently close to x_p such that

$$x''_{p-1} < x'_p = x''_p < x_{p+1}$$

and

$$|u(x'_p) - u(x_p)| \leq \frac{\varepsilon}{P}.$$

Notice that x''_{p-1} is already defined when we define x'_p and x''_p.

(2) If x_p is a jump point of u, then take $x'_p \le x_p \le x''_p$ in $I \setminus E$ such that

$$\begin{cases} u(x'_p) \le u(x_p) \le u(x''_p) & \text{if } \lim_{x\to x_p^-} u(x) < \lim_{x\to x_p^+} u(x), \\ u(x'_p) \ge u(x_p) \ge u(x''_p) & \text{if } \lim_{x\to x_p^-} u(x) > \lim_{x\to x_p^+} u(x). \end{cases}$$

We can now perform the following computation:

$$\begin{aligned}
&\sum_{p=1}^{P-1} \big|u(x_{p+1}) - u(x_p)\big| \\
&\qquad \le \sum_{p=1}^{P-1} |u(x_{p+1}) - u(x'_{p+1})| + |u(x'_{p+1}) - u(x''_p)| + |u(x''_p) - u(x_p)| \\
&\qquad = \sum_{p=2}^{P} |u(x_p) - u(x'_p)| + \sum_{p=1}^{P-1} |u(x'_{p+1}) - u(x''_p)| + \sum_{p=1}^{P-1} |u(x''_p) - u(x_p)| \\
&\qquad = \sum_{p=1}^{P} \Big[|u(x_p) - u(x'_p)| + |u(x''_p) - u(x_p)|\Big] + \sum_{p=1}^{P-1} |u(x'_{p+1}) - u(x''_p)| \\
&\qquad = \sum_{\substack{p \text{ contin.}\\ \text{point}}} \frac{2\varepsilon}{P} + \sum_{\substack{p \text{ jump}\\ \text{point}}} \Big[|u(x_p) - u(x'_p)| + |u(x''_p) - u(x_p)|\Big] + \sum_{p=1}^{P-1} |u(x'_{p+1}) - u(x''_p)| \\
&\qquad \le 2\varepsilon + \sum_{\substack{p \text{ jump}\\ \text{point}}} |u(x''_p) - u(x'_p)| + \sum_{p=1}^{P-1} |u(x'_{p+1}) - u(x''_p)| \\
&\qquad \le 2\varepsilon + \sum_{p=1}^{P} |u(x''_{p+1}) - u(x'_{p+1})| + |u(x'_{p+1}) - u(x''_p)| \\
&\qquad \le 2\varepsilon + \sup\left\{ \sum_{p=1}^{P-1} \big|u(x_{p+1}) - u(x_p)\big| \;\middle|\; P \in \mathbb{N}\ x_p \in I \setminus E \text{ for any } p \text{ and } x_1 < \cdots < x_P \right\}
\end{aligned}$$

and thus by the arbitrarity of $\varepsilon > 0$

$$\text{p.Tot.Var.}(u; I) \le \sup\left\{ \sum_{p=1}^{P-1} \big|u(x_{p+1}) - u(x_p)\big| \;\middle|\; P \in \mathbb{N}\ x_p \in I\setminus E \text{ for any } p \text{ and } x_1 < \cdots < x_P \right\},$$

thus concluding the proof of the lemma. □

Corollary 1.38. *For any map $v : I \to \mathbb{R}$, even if v is not a good representative, and for any $E \subseteq I$ such that $\mathcal{L}^1(E) = 0$, it holds*

$$\text{e.Tot.Var.}(v; I) \le \text{p.Tot.Var.}(v; I \setminus E).$$

Proof. Let u be a good representative for e.Tot.Var.$(v; I)$. Let $F := \{u \neq v\}$. Clearly $\mathcal{L}^1(F) = 0$. It holds

$$\begin{aligned}
\text{e.Tot.Var.}(v;I) &= \text{e.Tot.Var.}(u;I) \\
\text{(since } u \text{ is a good representative)} &= \text{p.Tot.Var.}(u;I) \\
\text{(by previous lemma and the fact that } \mathcal{L}^1(E\cup F)=0) &= \text{p.Tot.Var.}(u;I\setminus(E\cup F)) \\
\text{(since } u=v \text{ on } I\setminus(E\cup F)) &= \text{p.Tot.Var.}(v;I\setminus(E\cup F)) \\
&\leq \text{p.Tot.Var.}(v;I\setminus E),
\end{aligned}$$

thus concluding the proof of the corollary. □

The next theorem is the well-known compactness property of the BV functions.

Theorem 1.39. *Let $I \subseteq \mathbb{R}$ be an interval. Let $\mathcal{F} \subseteq L^1(I)$ be a family of functions such that*

$$\sup\left\{ \|u\|_{L^1(I)} + \text{e.Tot.Var.}(u;I) \;\middle|\; u \in \mathcal{F} \right\} < \infty.$$

Then $\mathcal{F}$ is pre-compact in L^1.

For a proof, see [**AFP00**, Theorem 3.23].

1.5. Monotone multi-functions

We conclude this chapter with a very short presentation of monotone multi-functions in one variable and we prove some related results.

Definition 1.40. Let $I \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$. Let $\mu \ll \mathcal{L}^1$ be a measure on I a.c. w.r.t. the Lebesgue measure. Let $\Phi : I \to 2^{\mathbb{R}}$ be a (multi-valued) function from I to $\mathbb{R}$, defined for a.e. $x \in I$. We say that Φ is a *(monotone) increasing multi-function* (w.r.t. the measure μ) if for $\mu \otimes \mu$-a.e. $(z, z') \in I^2$ and for any $x, x' \in \mathbb{R}$

$$\text{if } x \in \Phi(z), x' \in \Phi(z'), \text{ then } (x - x')(z - z') \geq 0.$$

A similar definition holds for *(monotone) decreasing multi-functions.*

Lemma 1.41. *Let $I \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$. Let $\Phi : I \to 2^{\mathbb{R}}$ be a monotone increasing multi-function wr.t. the Lebesgue measure on I, as in the previous definition. Then*

(1) Φ is single-valued for up to a countable number of $z \in I$;
(2) there exists an increasing map $\tilde{\Phi} : I \to \mathbb{R}$ such that

$$\{\tilde{\Phi}(z)\} = \Phi(z) \text{ up to a countable number of } z \in I;$$

(3) if we require that $\tilde{\Phi}$ is left-continuous, such map is unique.

Proof. Let $z \in I$ such that $\Phi(z)$ is not single-valued. Then we can find a rational number $q(z) \in \big(\inf \Phi(z), \sup \Phi(z)\big)$. The map $z \mapsto q(z)$ is injective: indeed, if $z < z'$, then it must be $\sup \Phi(z) < \inf \Phi(z')$ and thus $q(z) < q(z')$. Therefore there exist at most a countable number of $z \in I$ such that $\Phi(z)$ is not single-valued. The second and third point follow easily from the first one. □

Thanks to previous lemma, we can always identify a monotone increasing multi-valued function with its unique left-continuous representative.

Proposition 1.42. *A family $\mathcal{F} \subseteq L^1(I;\mu)$ of monotone increasing functions (w.r.t. the measure μ) over an interval $I \subseteq \mathbb{R}$ is compact in $L^1(I;\mu)$.*

PROOF. The proof is an easy consequence of the compactness criterion in L^1 and the definition of monotone increasing function. $\square$

DEFINITION 1.43. Let $\Phi : I \to 2^{\mathbb{R}}$ be a monotone increasing multi-function. Set $\Phi(I) := \bigcup\{\phi(x) \mid x \in I\}$. The *pseudo-inverse* of Φ is the multi-function $\Phi^{-1} : \Phi(I) \to 2^{\mathbb{R}}$ defined as $\Phi^{-1}(y) := \{x \in I \mid y \in \Phi(x)\}$.

It is immediate to see that if Φ is a monotone increasing (resp. decreasing) multi-function, then Φ^{-1} is a monotone increasing (resp. decreasing) multi-function.

CHAPTER 2

Preliminaries on conservation laws

In this chapter we introduce some notations and results in the theory of Hyperbolic Conservation Laws. This is not an extensive discussion of the general theory. On the contrary, it is just a collection of those objects and statements which will be widely use in the subsequent chapters.

In Section 2.1 we show how an entropic self-similar solution to the Riemann problem (u^L, u^R) is constructed, focusing our attention especially on the proof of the existence of the elementary curves of a fixed family. Even if the main ideas are similar to the standard proof found in the literature (see for instance [BB05]), we need to use a slightly different distance among elementary curves, see (2.11): this because we need sharper estimate on the variation of speed.

In Section 2.2 we recall the definitions of some quantities which in some sense measure how strong the interaction is between two contiguous Riemann problems which are joining and we present some related results: these quantities are the *transversal amount of interaction* (Definition 2.5), the *cubic amount of interaction* (Definition 2.6), the *amount of cancellation* (Definition 2.8) and the *amount of creation* (Definition 2.8).

In Section 2.3 we review how a family of approximate solutions $\{u^\varepsilon(t,x)\}_{\varepsilon>0}$, to the Cauchy problem

$$\begin{cases} u_t + F(u)_x = 0, \\ u(0,x) = \bar{u}(x), \end{cases} \tag{2.1}$$

is constructed by means of the Glimm scheme.

Finally in Section 2.4 we recall the definitions of some Lyapunov functionals, already present in the literature, which provide a uniform-in-time bound on the spatial total variation of the approximate solution u^ε.

Recall that $F : \Omega \to \mathbb{R}^N$ is a smooth (say C^3) function, defined on a neighborhood Ω of a compact set K, satisfying the strict hyperbolicity condition, i.e. the Jacobian matrix $A(u) := Df(u)$ has N real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$. W.l.o.g. we assume also that $0 \in K$ and that $\lambda_k(u) \in [0,1]$ for any u. This can always be achieved through a linear change of variable. Moreover, since we will consider only solutions with small total variation taking values in K, we will also assume that all the derivatives of f are bounded on Ω and that there exist constant $\hat{\lambda}_0, \dots, \hat{\lambda}_n$ such that

$$0 < \hat{\lambda}_{k-1} < \lambda_k(u) < \hat{\lambda}_k < 1, \qquad \text{for every } u \in \Omega,\ k = 1, \dots, n. \tag{2.2}$$

2.1. Vanishing viscosity solution to the Riemann problem

We describe here the method developed in [BB05], with some minimal variations, to construct a solution to the Riemann problem (u^L, u^R), i.e. the system (2.1) together with the initial datum

$$u(0,x) = \begin{cases} u^L, & x \geq 0, \\ u^R, & x < 0, \end{cases} \tag{2.3}$$

provided that $|u^R - u^L|$ is small enough. First we present the algorithm used to build the solution to the Riemann problem (u^L, u^R) and then we focus our attention on the construction of the elementary curves of a fixed family.

2.1.1. Algorithm for solving the Riemann problem. The following proposition holds.

PROPOSITION 2.1. *For all $\delta_2 > 0$ there exists $0 < \delta_1 < \delta_2$ such that for any $u^L, u^R \in B(0, \delta_1)$ the Riemann problem (2.1), (2.3) admits a unique, self-similar, right continuous, vanishing viscosity solution, taking values in $B(0, \delta_2)$.*

SKETCH OF THE PROOF. *Step 1.* For any index $k \in \{1, \dots, n\}$, through a Center Manifold technique, one can find a neighborhood of the point $(0, 0, \lambda_k(0))$ of the form

$$\mathcal{D}_k := \Big\{(u_k, v_k, \sigma_k) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \;\Big|\; |u_k| \le \rho, |v_k| \le \rho, |\sigma_k - \lambda_k(0)| \le \rho\Big\} \tag{2.4}$$

for some $\rho > 0$ (depending only on f) and a smooth vector field

$$\tilde{r}_k : \mathcal{D}_k \to \mathbb{R}^N, \qquad \tilde{r}_k = \tilde{r}_k(u_k, v_k, \sigma_k),$$

satisfying

$$\tilde{r}_k(u_k, 0, \sigma_k) = r_k(u_k), \qquad \left|\frac{\partial \tilde{r}_k}{\partial \sigma_k}(u_k, v_k, \sigma_k)\right| \le \mathcal{O}(1)|v_k|. \tag{2.5}$$

Here and in the following, the index k is just to remind that we are working with curves of the k-th fixed family. We will call $\tilde{r}_k$ the *k-generalized eigenvector*. The characterization of $\tilde{r}_k$ is that

$$\mathcal{D}_k \ni (u_k, v_k, \sigma_k) \mapsto \big(u_k, v_k\tilde{r}_k, \sigma_k\big) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$$

is a parameterization of a center manifold near the equilibrium $(0, 0, \lambda_k(0)) \in \mathcal{D}_k$ for the ODE of traveling waves

$$\big(A(u) - \sigma\mathbb{I}\big)u_x = u_{xx} \qquad \Longleftrightarrow \qquad \begin{cases} u_x = v, \\ v_x = (A(u) - \sigma\mathbb{I})v, \\ \sigma_x = 0, \end{cases}$$

where $A(u) = Df(u)$, the Jacobian matrix of the flux f, and $\mathbb{I}$ is the identity $N \times N$ matrix.

Associated to the generalized eigenvectors, we can define smooth functions $\tilde{\lambda}_k : \mathcal{D}_k \to \mathbb{R}$ by

$$\tilde{\lambda}_k(u_k, v_k, \sigma_k) := \big\langle l_k(u_k), A(u_k)\tilde{r}_k(u_k, v_k, \sigma_k)\big\rangle.$$

We will call $\tilde{\lambda}_k$ the *k-generalized eigenvalue*. By (2.5) and the definition of $\tilde{\lambda}_k$, we can get

$$\tilde{\lambda}_k(u_k, 0, \sigma_k) = \lambda_k(u_k), \qquad \left|\frac{\partial \tilde{\lambda}_k}{\partial \sigma_k}(u_k, v_k, \sigma_k)\right| \le \mathcal{O}(1)|v_k|. \tag{2.6}$$

For the construction of the generalized eigenvectors and eigenvalues and the proof of (2.5), (2.6), see Section 4 of [BB05].

Step 2. By a fixed point technique one can now prove that there exist $\delta, \eta > 0$ (depending only on f), such that for any

$$k \in \{1, \dots, n\}, \qquad u^L \in B(0, \rho/2), \qquad 0 < s < \eta,$$

there is a curve

$$\begin{array}{rccc} \gamma_k & : & [0, s] & \to & \mathcal{D}_k \\ & & \tau & \mapsto & \gamma_k(\tau) = (u_k(\tau), v_k(\tau), \sigma_k(\tau)) \end{array}$$

such that $u_k, v_k \in C^{1,1}([0,s])$, $\sigma_k \in C^{0,1}([0,s])$, it takes values in $B(u^L,\delta) \times B(0,\delta) \times B(\lambda_k(u^L),\delta)$ and it is the unique solution to the system

$$\begin{cases} u_k(\tau) = u^L + \displaystyle\int_0^\tau \tilde{r}_k(\gamma_k(\varsigma))d\varsigma \\ v_k(\tau) = f_k(\gamma_k;\tau) - \operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau) \\ \sigma_k(\tau) = \dfrac{d}{d\tau} \operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau) \end{cases} \tag{2.7}$$

where

$$f_k(\gamma_k;\tau) := \int_0^\tau \tilde{\lambda}_k(\gamma_k(\varsigma))d\varsigma. \tag{2.8}$$

and $\operatorname{conv}_{[0,s]} f_k$ is the convex envelope of f_k in the interval $[0,s]$, see Definition 1.1. In the case $s < 0$ a completely similar result holds, replacing the convex envelope with the concave one. If $s = 0$, we assume that the curve $\gamma_k : \{0\} \to \mathcal{D}$ is made by one single point, $\gamma_k(0) = (u^L, 0, \lambda_k(u^L))$.
If we want to stress the dependence of the curve γ_k on k, u^L and s we will use the notation

$$\gamma_k = \Gamma_k(u^L,s) = \Big(u_k(u^L,s)(\tau), v_k(u^L,s)(\tau), \sigma_k(u^L,s)(\tau)\Big).$$

The curve $\Gamma_k(u^L,s)$ will be called the *exact curve* of the k-th family with length s and starting point u^L.

Even if the existence and uniqueness of such a curve is known, we give a proof of this fact in Section 2.1.2, since we need to use a definition of distance among curves slightly different from the one in [BB05].

Step 3. Once the curve γ_k solving (2.7) is found, one can prove the following lemma.

LEMMA 2.2. *Let $\gamma_k : [0,s] \to \mathcal{D}_k$, $\gamma_k = \Gamma_k(u^L,s) = (u_k(\tau), v_k(\tau), \sigma_k(\tau))$, be the Lipschitz curve solving the system (2.7) and define the right state $u^R := u_k(s)$. Then the unique, right continuous, vanishing viscosity solution of the Riemann problem (u^L,u^R) is the function*

$$\omega(t,x) := \begin{cases} u^L & \text{if } x/t \le \sigma_k(0), \\ u_k(\tau) & \text{if } x/t = \max\{\xi \in [0,s] \mid x/t = \sigma_k(\xi)\}, \\ u^R & \text{if } x/t \ge \sigma_k(s). \end{cases}$$

For the proof see Lemma 14.1 in [BB05]. The case $s < 0$ is completely similar.

Step 4. By previous step, for any $k \in \{1,\dots,n\}$, $u^L \in B(0,\rho/2)$, there is a curve

$$(-\eta,\eta) \ni s \quad \mapsto \quad T_s^k(u^L) := u_k(u^L,s)(s) \in B(u^L,\delta) \subseteq \mathbb{R}^N$$

such that the Riemann problem $(u^L, T_s^k(u^L))$ admits a self similar solution consisting only of k-waves.

LEMMA 2.3. *The curve $s \mapsto T_s^k(u^L)$ is Lipschitz continuous and*

$$\operatorname*{ess\,lim}_{s\to 0} \frac{dT_s^k(u^L)}{ds} = r_k(u^L). \tag{2.9}$$

For the proof see Lemma 14.3 in [BB05].

Step 5. Thanks to (2.9), the solution to the general Riemann problem (2.1), (2.3) can be now constructed following a standard procedure (see for example [Daf05, Chapter 9]). One considers the composite map

$$\begin{array}{rcccl} T(u^L) & : & (-\eta,\eta)^N & \to & \mathbb{R}^N \\ & & (s_1,\dots,s_N) & \mapsto & T(u^L)(s_1,\dots,s_N) := T^N_{s_N}\circ\cdots\circ T^1_{s_1}(u^L) \end{array}$$

By (2.9) and a version of the Implicit Function Theorem valid for Lipschitz continuous maps, $T(u^L)$ is a one-to-one mapping from a neighborhood of the origin in $\mathbb{R}^N$ onto a neighborhood of u^L. Hence, for all u^R sufficiently close to u^L (uniformly w.r.t. $u^L\in B(0,\rho/2)$), one can find unique values $s_1,\dots,s_N$ such that $T(u^L)(s_1,\dots,s_N)=u^R$.
In turn, this yields intermediate states $u_0=u^L,u_1,\dots,u_N=u^R$ such that each Riemann problem with data (u_{k-1},u_k) admits a vanishing viscosity solution $\omega_k=\omega_k(t,x)$ consisting only of k-waves, i.e.

$$u^R = T^N_{s_N}\circ\cdots\circ T^1_{s_1}u^L \tag{2.10}$$

for some $(s_1,\dots,s_N)\in\mathbb{R}^N$. Observe that some of the s_k, $k=1,\dots,N$, can be equal to zero. By the assumption (2.2) we can now define the solution to the general Riemann problem (u^L,u^R) by

$$\omega(t,x)=\omega_k(t,x) \qquad \text{for } \hat\lambda_{k-1}<\frac{x}{t}<\hat\lambda_k.$$

Therefore we can choose $\delta_1,\delta_2\ll 1$ such that if $u^L,u^R\in B(0,\delta_1)$, the Riemann problem (u^L,u^R) can be solved as above and the solution takes values in $B(0,\delta_2)$, thus concluding the proof of the proposition. □

2.1.2. Proof of Step 2. We now explicitly prove that the system (2.7) admits a $C^{1,1}\times C^{1,1}\times C^{0,1}$-solution, i.e. we prove Step 2 of the previous algorithm, using the Contraction Mapping Principle. As we said, we need a proof slightly different from the one in [BB05]: in fact, even if the general approach is the same, the distance used among curves is suited for the type of estimates we are interested in.

Fix an index $k=1,\dots,N$ and consider the space

$$X := L^\infty\big([0,s];\mathbb{R}^N\big)\times L^\infty\big([0,s]\big)\times L^1\big([0,s]\big)$$

A generic element of X will be denoted by $\gamma_k=(u_k,v_k,\sigma_k)$. The index k is just to remember that we are solving a RP with wavefronts of the k-th family. Endow X with the norm

$$\|\gamma_k\|_\dagger = \big\|(u_k,v_k,\sigma_k)\big\|_\dagger := \|u_k\|_\infty+\|v_k\|_\infty+\|\sigma_k\|_1 \tag{2.11}$$

and consider the subset

$$\begin{aligned} X_k(u^L,s) := \Big\{\gamma_k=(u_k,v_k,\sigma_k)\in X : {} & u_k,v_k \text{ are Lipschitz and } \mathrm{Lip}(u_k)+\mathrm{Lip}(v_k)\le L, \\ & u_k(0)=u^L, v_k(0)=0, \\ & |u_k(\tau)-u^L|\le\delta, |v_k(\tau)|\le\delta \text{ for any } \tau\in[0,s], \\ & |\sigma_k(\tau)-\lambda_k(u^L)|\le\delta \text{ for } \mathcal{L}^1\text{-a.e. } \tau\in[0,s]\Big\} \end{aligned}$$

for $u^L\in B(0,\rho)$ and $L,s,\delta>0$ which will be chosen later. Clearly $X_k(u^L,s)$ is a closed subset of the Banach space X and thus it is a complete metric space. Denote by D the distance induced by the norm $\|\cdot\|_\dagger$ on X.

Consider now the transformation

$$\begin{array}{rcccl} \mathcal{T} & : & X_k(u^L,s) & \to & X \\ & & \gamma_k & \mapsto & \hat\gamma_k := \mathcal{T}\gamma_k \end{array}$$

defined by the formula

$$\begin{cases} \hat{u}_k(\tau) := u^L + \displaystyle\int_0^\tau \tilde{r}_k(\gamma_k(\varsigma))d\varsigma, \\ \hat{v}_k(\tau) := f_k(\gamma_k;\tau) - \operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau), \\ \hat{\sigma}_k(\tau) := \dfrac{d}{d\tau}\operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau), \end{cases}$$

where f_k has been defined in (2.8). Observe that, since $\tilde{\lambda}_k$ is uniformly bounded near $(u^L, 0, \lambda_k(u^L))$, it turns out that $f_k(\gamma_k)$ is a Lipschitz function for any $\gamma_k \in \Gamma_k(u^L, s)$, and thus by Theorem 1.3, Point (1), $\operatorname*{conv}_{[0,s]} f_k(\gamma_k) : [0,s] \to \mathbb{R}$ is Lipschitz and its derivative is in $L^\infty([0,s],\mathbb{R})$.

Lemma 2.4. *There exist $L, \eta, \delta > 0$ depending only on f such that for all fixed $u^L \in B(0,\rho/2)$ it holds:*

(1) for any $|s| < \eta$, $\mathcal{T}$ is a contraction from $\Gamma_k(u^L,s)$ into itself, more precisely

$$\big\|\mathcal{T}(\gamma_k) - \mathcal{T}(\gamma_k')\big\|_\dagger \le \frac{1}{2}\big\|\gamma_k - \gamma_k'\big\|_\dagger;$$

(2) if $\bar{\gamma}_k = (\bar{u}_k, \bar{v}_k, \bar{\sigma}_k)$ is the fixed point of $\mathcal{T}$, then $\bar{u}_k, \bar{v}_k \in C^{1,1}$ and $\bar{\sigma}_k \in C^{0,1}$.

Clearly Point (2) above yields Step 2 of the proof of Proposition 2.1.

Proof. *Step 1.* We first prove that if $\gamma_k \in X_k(u^L, s)$, then $\hat{\gamma}_k = (\hat{u}_k, \hat{v}_k, \hat{\sigma}_k) := \mathcal{T}(\gamma_k) \in X_k(u^L, s)$, provided $L \gg 1$, $\eta \ll 1$, while δ will be fixed in the next step.

Clearly $\hat{u}_k(0) = u^L$ and $\hat{v}_k(0) = 0$. Moreover $\hat{u}_k, \hat{v}_k$ are Lipschitz continuous and $\hat{\sigma}_k$ is in $L^\infty([0,s])$.

Let us prove the uniform estimate on the Lipschitz constants. First we have

$$\big|\hat{u}_k(\tau_2) - \hat{u}_k(\tau_1)\big| \le \int_{\tau_1}^{\tau_2} \big|\tilde{r}_k(\gamma_k(\varsigma))\big| d\varsigma \le \|\tilde{r}_k\|_\infty |\tau_2 - \tau_1| \le \frac{L}{2}\big|\tau_2 - \tau_1\big|.$$

if the constant L is big enough. For v_k it holds

$$\begin{aligned} \big|\hat{v}_k(\tau_2) - \hat{v}_k(\tau_1)\big| &\le \big|f_k(\gamma_k;\tau_2) - f_k(\gamma_k;\tau_1)\big| + \Big|\operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau_2) - \operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau_1)\Big| \\ \text{(by Theorem 1.3, Point (1))} &\le 2 \operatorname{Lip}\big(f_k(\gamma_k)\big)\big|\tau_2 - \tau_1\big| \le 2\,\big\|\tilde{\lambda}_k\big\|_\infty \big|\tau_2 - \tau_1\big| \\ &\le \frac{L}{2}\big|\tau_2 - \tau_1\big|, \end{aligned}$$

if L is big enough.

Finally let us prove that the curve γ_k remains uniformly close to the point $(u^L, 0, \lambda_k(u^L))$. First we have

$$\big|\hat{u}_k(\tau) - u^L\big| \le \int_0^\tau \big|\tilde{r}_k(\gamma_k(\varsigma))\big| d\varsigma \le \|\tilde{r}_k\|_\infty |\tau| \le \|\tilde{r}_k\|_\infty |\eta| \le \delta,$$

if $\eta \ll 1$. Next it holds

$$
\begin{aligned}
\big|\hat v_k(\tau)\big| &\le \big|f_k(\gamma_k;\tau)\big| + \big|\operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau)\big| \\
&\le \int_0^\tau \left|\frac{df_k(\gamma_k;\varsigma)}{d\varsigma}\right| d\varsigma \ + \ \int_0^\tau \left|\frac{d}{d\varsigma}\operatorname*{conv}_{[0,s]} f_k(\gamma_k;\varsigma)\right| d\varsigma \\
\text{(by Proposition 1.11)} &\le 2\int_0^\tau \big|(\tilde\lambda_k\circ\gamma_k)(\varsigma)\big| d\varsigma \\
&\le \|\tilde\lambda_k\|_\infty |\eta| \le \delta,
\end{aligned}
$$

if $\eta \ll 1$. Finally, before making the computation for σ_k, let us observe that

$$
\begin{aligned}
\left|\frac{df_k(\gamma_k;\tau)}{d\tau} - \lambda_k(u^L)\right| &\le \Big|\tilde\lambda_k\big(u_k(\tau),v_k(\tau),\sigma_k(\tau)\big) - \tilde\lambda_k\big(u^L,0,\sigma_k(\tau)\big)\Big| \\
&\le \mathcal{O}(1)\Big(\big|u_k(\tau)-u^L\big| + \big|v_k(\tau)\big|\Big) \\
&\le \mathcal{O}(1)\Big(\mathrm{Lip}(u_k) + \mathrm{Lip}(v_k)\Big)\eta \\
&\le \mathcal{O}(1)L\eta \le \delta,
\end{aligned}
$$

if $\eta \ll 1$. Hence

$$
\begin{aligned}
\big\|\hat\sigma_k - \lambda_k(u^L)\big\|_\infty &= \left\|\frac{d}{d\tau}\operatorname*{conv}_{[0,s]} f_k(\gamma_k) - \lambda_k(u^L)\right\|_\infty \\
\text{(by Prop. 1.11)} &\le \left\|\frac{df_k(\gamma_k)}{d\tau} - \lambda_k(u^L)\right\|_\infty \\
&\le \delta,
\end{aligned}
$$

if $\eta \ll 1$.

We have thus proved that we can choose $L \gg 1$, $\eta \ll 1$ such that $\hat\gamma_k := \mathcal{T}(\gamma_k) \in X_k(u^L,s)$. Notice that the choice of L,η depends only on f and δ and not on $u^L \in B(0,\delta/2)$.

Step 2. We now prove that the map $\mathcal{T} : X_k(u^L,s) \to X_k(u^L,s)$ is a contraction. Let $\gamma_k = (u_k,v_k,\sigma_k), \gamma_k' = (u_k',v_k',\sigma_k') \in X_k(u^L,s)$ and set

$$
\hat\gamma_k = (\hat u_k,\hat v_k,\hat\sigma_k) := \mathcal{T}(\gamma_k), \qquad \hat\gamma_k' = (\hat u_k',\hat v_k',\hat\sigma_k') := \mathcal{T}(\gamma_k').
$$

It holds for the component u_k

$$
\begin{aligned}
\big|\hat u_k(\tau) - \hat u_k'(\tau)\big| &\le \int_0^\tau \big|\tilde r_k(\gamma_k(\varsigma)) - \tilde r_k(\gamma_k'(\varsigma))\big| d\varsigma \\
&\le \int_0^\tau \left(\left\|\frac{\partial \tilde r_k}{\partial u_k}\right\|_\infty \big|u_k(\varsigma)-u_k'(\varsigma)\big| + \left\|\frac{\partial \tilde r_k}{\partial v_k}\right\|_\infty \big|v_k(\varsigma)-v_k'(\varsigma)\big| + \left\|\frac{\partial \tilde r_k}{\partial \sigma_k}\right\|_\infty \big|\sigma_k(\varsigma)-\sigma_k'(\varsigma)\big|\right) d\varsigma \\
\text{(by (2.5))} &\le \mathcal{O}(1)\int_0^\tau \Big(\big|u_k(\varsigma)-u_k'(\varsigma)\big| + \big|v_k(\varsigma)-v_k'(\varsigma)\big| + \delta\big|\sigma_k(\varsigma)-\sigma_k'(\varsigma)\big|\Big) d\varsigma \\
&\le \mathcal{O}(1) D(\gamma_k,\gamma_k')(\eta\ +\delta) \\
&\le \frac{1}{2} D(\gamma_k,\gamma_k'),
\end{aligned}
\tag{2.12}
$$

if $\eta,\delta \ll 1$.

For the component v_k we have

$$\begin{aligned}
\big|\hat{v}_k(\tau) - \hat{v}'_k(\tau)\big| &\le \big|f_k(\gamma_k;\tau) - f_k(\gamma'_k;\tau)\big| + \big|\operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau) - \operatorname*{conv}_{[0,s]} f_k(\gamma'_k;\tau)\big| \\
\text{(by Proposition 1.11)} &\le 2\big\|f_k(\gamma_k) - f_k(\gamma'_k)\big\|_\infty \\
&\le 2\left\|\frac{df_k(\gamma_k)}{d\tau} - \frac{df_k(\gamma'_k)}{d\tau}\right\|_1 \\
&= 2\int_0^s \Big|\tilde{\lambda}_k\big(\gamma_k(\tau)\big) - \tilde{\lambda}_k\big(\gamma'_k(\tau)\big)\Big| d\tau \\
\text{(using (2.6) as in (2.12))} &\le \frac{1}{2} D(\gamma_k, \gamma'_k),
\end{aligned} \tag{2.13}$$

if $\eta, \delta \ll 1$.

Finally

$$\big\|\hat{\sigma}_k - \hat{\sigma}'_k\big\|_1 \le \int_0^s \Big|\tilde{\lambda}_k\big(\gamma_k(\varsigma)\big) - \tilde{\lambda}_k\big(\gamma'_k(\tau)\big)\Big| \le \frac{1}{2} D(\gamma_k, \gamma'_k),$$

if $\eta, \delta \ll 1$ using (2.6) as in (2.13).

Hence $\mathcal{T}$ is a contraction from $X_k(u^L, s)$ into itself, with contractive constant equal to $1/2$, provided $\eta, \delta \ll 1$.

Step 3. Let us now prove the second part of the lemma, concerning the regularity of the fixed point $\bar{\gamma}_k = (\bar{u}, \bar{v}_k, \bar{\sigma}_k)$. It is immediate to see that $\bar{u}, \bar{v} \in C^{1,1}$.

Fix now a big constant $M > 0$ and let $A(M) \subseteq X_k(u^L, s)$ be the subset which contains all the curves $\gamma_k = (u_k, v_k, \sigma_k)$ such that σ_k is Lipschitz with $\mathrm{Lip}(\sigma_k) \le M$. Clearly $A(M)$ is non empty and closed in X. We claim that $\mathcal{T}(A(M)) \subseteq A(M)$ if M is big enough and δ, η small enough. This will conclude the proof of the lemma.

Let $\gamma_k \in A(M)$ and, as before, $\hat{\gamma}_k = (\hat{u}_k, \hat{v}_k, \hat{\sigma}_k) := \mathcal{T}(\gamma_k)$. Let us first compute the Lipschitz constant of $\frac{df_k(\gamma_k)}{d\tau}$:

$$\begin{aligned}
\left|\frac{df_k(\gamma_k;\tau_2)}{d\tau} - \frac{df_k(\gamma_k;\tau_1)}{d\tau}\right| &= \Big|\tilde{\lambda}_k(\gamma_k(\tau_2)) - \tilde{\lambda}_k(\gamma_k(\tau_1))\Big| \\
&\le \mathcal{O}(1)\Big|\Big(\mathrm{Lip}(u_k) + \mathrm{Lip}(v_k) + \delta \mathrm{Lip}(\sigma_k)\Big)\Big||\tau_2 - \tau_1| \\
&\le \mathcal{O}(1)(2L + \delta M)\big|\tau_2 - \tau_1\big| \\
&\le M\big|\tau_2 - \tau_1\big|,
\end{aligned}$$

if $0 < \delta \ll 1$ and $M \gg 1$. Now observe that

$$\begin{aligned}
\big|\hat{\sigma}_k(\tau_2) - \hat{\sigma}_k(\tau_1)\big| &\le \left|\frac{d}{d\tau}\operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau_2) - \frac{d}{d\tau}\operatorname*{conv}_{[0,s]} f_k(\gamma_k;\tau_1)\right| \\
&\le \mathrm{Lip}\left(\frac{d}{d\tau}\operatorname*{conv}_{[0,s]} f_k(\gamma_k)\right)\big|\tau_2 - \tau_1\big| \\
\text{(by Theorem 1.3, Point (4))} &\le \mathrm{Lip}\left(\frac{df_k(\gamma_k)}{d\tau}\right)\big|\tau_2 - \tau_1\big| \\
&\le M\big|\tau_2 - \tau_1\big|.
\end{aligned}$$

Hence $\hat{\sigma}$ is Lipschitz and $\mathrm{Lip}(\hat{\sigma}) \le M$, i.e. $\hat{\gamma}_k \in A(M)$, if $\delta \ll 1$ and $M \gg 1$. $\square$

2.2. Definition of the amounts of transversal interaction, cancellation and creation

In this section we introduce some quantities, namely the *transversal amount of interaction* (Definition 2.5), the *cubic amount of interaction* (Definition 2.6), the *amount of cancellation* (Definition 2.8) and the *amount of creation* (Definition 2.8), which measure how strong the interaction is between two contiguous Riemann problems and we present some related results. All these quantities are already present in the literature. In Section 3.2.2, we will introduce one more quantity, which will be called the *quadratic amount of interaction*, introduced for the first time in [**BM15b**].

Consider two contiguous Riemann problem, whose resolution in elementary waves is

$$u^M = T^N_{s'_N} \circ \cdots \circ T^1_{s'_1} u^L, \qquad u^R = T^N_{s''_N} \circ \cdots \circ T^1_{s''_1} u^M, \tag{2.14}$$

and the Riemann problem obtained by joining them, resolved by

$$u^R = T^N_{s_N} \circ \cdots \circ T^1_{s_1} u^L.$$

Let f'_k, f''_k be the two reduced fluxes of the k-th waves s'_k, s''_k for the Riemann problems (u^L, u^M), (u^M, u^R), respectively: more precisely, f'_k (f''_k) is computed by (2.8) where, for $k = 1, \dots, n$, γ'_k (γ''_k) is the solutions of (2.7) with length s'_k (s''_k) and initial point

$$u^L \text{ for } k = 1, \quad T^{k-1}_{s'_{k-1}} \circ \cdots \circ T^1_{s'_1} u^L, \text{ for } k \geq 2,$$

$$\left(u^M \text{ for } k = 1, \quad T^{k-1}_{s''_{k-1}} \circ \cdots \circ T^1_{s''_1} u^M, \text{ for } k \geq 2. \right)$$

Since (2.7) is invariant when we add a constant to f_k, having in mind to perform the merging operations (1.2), (1.3), we can assume that f''_k is defined on $s'_k + \mathbf{I}(s''_k)$ and satisfies $f''_k(s'_k) = f'_k(s'_k)$.

Definition 2.5. The quantity

$$\mathtt{A}^{\text{trans}}(u^L, u^M, u^R) := \sum_{1 \leq h < k \leq n} |s'_k||s''_h|$$

is called the *transversal amount of interaction* associated to the two Riemann problems (2.14).

Definition 2.6. For $s'_k > 0$, we define *cubic amount of interaction of the k-th family* for the two Riemann problems (u^L, u^M), (u^M, u^R) as follows:

(1) if $s''_k \geq 0$,

$$\begin{aligned} \mathtt{A}^{\text{cubic}}_k(u^L, u^M, u^R) := & \int_0^{s'_k} \Big[\operatorname*{conv}_{[0, s'_k]} f'_k(\tau) - \operatorname*{conv}_{[0, s'_k + s''_k]} (f'_k \cup f''_k)(\tau) \Big] d\tau \\ & + \int_{s'_k}^{s'_k + s''_k} \Big[\operatorname*{conv}_{[s'_k, s''_k]} f''_k(\tau) - \operatorname*{conv}_{[0, s'_k + s''_k]} (f'_k \cup f''_k)(\tau) \Big] d\tau; \end{aligned}$$

(2) if $-s'_k \leq s''_k < 0$

$$\begin{aligned} \mathtt{A}^{\text{cubic}}_k(u^L, u^M, u^R) := & \int_0^{s'_k + s''_k} \Big[\operatorname*{conv}_{[0, s'_k + s''_k]} f'_k(\tau) - \operatorname*{conv}_{[0, s'_k]} f'_k(\tau) \Big] d\tau \\ & + \int_{s'_k + s''_k}^{s'_k} \Big[\operatorname*{conc}_{[s'_k + s''_k, s'_k]} f'_k(\tau) - \operatorname*{conv}_{[0, s'_k]} f'_k(\tau) \Big] d\tau; \end{aligned}$$

(3) if $s_k'' < -s_k'$,

$$\mathtt{A}_k^{\text{cubic}}(u^L, u^M, u^R) := \int_{s_k'+s_k''}^{0} \Big[\operatorname*{conc}_{[s_k'+s_k'', s_k']} f_k''(\tau) - \operatorname*{conc}_{[s_k'+s_k'', 0]} f_k''(\tau) \Big] d\tau$$
$$+ \int_0^{s_k'} \Big[\operatorname*{conc}_{[s_k'+s_k'', s_k']} f_k''(\tau) - \operatorname*{conv}_{[0, s_k']} f_k''(\tau) \Big] d\tau.$$

Similar definitions can be given if $s_k' < 0$, interchanging convex envelopes with concave.

REMARK 2.7. The previous definition is exactly Definition 3.5 in [Bia03], where it is also shown that the terms appearing in the above definition are non negative.

The following definition is standard.

DEFINITION 2.8. The *amount of cancellation of the k-th family* is defined by

$$\mathtt{A}_k^{\text{canc}}(u^L, u^M, u^R) := \begin{cases} 0 & \text{if } s_k' s_k'' \geq 0, \\ \min\{|s_k'|, |s_k''|\} & \text{if } s_k' s_k'' < 0, \end{cases}$$

while the *amount of creation of the k-th family* is defined by

$$\mathtt{A}_k^{\text{cr}}(u^L, u^M, u^R) := \Big[|s_k| - |s_k' + s_k''| \Big]^+.$$

The following theorem is proved in [Bia03].

THEOREM 2.9. *It holds*

$$\sum_{k=1}^N \big| s_k - (s_k' + s_k'') \big| \leq \mathcal{O}(1) \Big[\mathtt{A}^{\text{trans}}(u^L, u^M, u^R) + \sum_{k=1}^N \mathtt{A}_k^{\text{cubic}}(u^L, u^M, u^R) \Big].$$

As an immediate consequence, we obtain the following corollary.

COROLLARY 2.10. *It holds*

$$\mathtt{A}_k^{\text{cr}}(u^L, u^M, u^R) \leq \mathtt{A}^{\text{trans}}(u^L, u^M, u^R) + \sum_{h=1}^N \mathtt{A}_h^{\text{cubic}}(u^L, u^M, u^R).$$

2.3. Construction of a Glimm approximate solution

We recall now briefly how an approximate solution $u^\varepsilon(t,x)$ to (2.1) is constructed by means of the Glimm scheme. Fix $\varepsilon > 0$.

To construct an approximate solution $u^\varepsilon = u^\varepsilon(t,x)$ to the Cauchy problem (2.1), we consider a grid in the (t,x) plane having step size $\Delta t = \Delta x = \varepsilon$, with nodes in the points

$$P_{i,m} = (t_i, x_m) := (i\varepsilon, m\varepsilon), \qquad i \in \mathbb{N}, \ m \in \mathbb{Z}.$$

Moreover we shall need a sequence of real numbers $\vartheta_1, \vartheta_2, \vartheta_3, \dots$. Now $\{\vartheta_i\}_i$ is any sequence of real numbers in $[0,1]$. It will be the topic of Chapter 4 to prove that if $\{\vartheta_i\}_i$ satisfies

$$\sup_{\lambda \in [0,1]} \left| \lambda - \frac{\operatorname{card}\{i \in \mathbb{N} \mid j_1 \leq i < j_2 \text{ and } \vartheta_i \in [0,\lambda]\}}{j_2 - j_1} \right| \leq C \cdot \frac{1 + \log(j_2 - j_1)}{j_2 - j_1}, \tag{2.15}$$

then the the Glimm approximations u^ε converges in L^1 at any fixed time $t \in [0,\infty)$ to the semigroup solution $S_t \bar{u}$ of the Cauchy problem (2.1) provided by Theorem 1.

At time $t = 0$, the Glimm algorithm starts by considering an approximation $\bar{u}^\varepsilon$ of the initial datum $\bar{u}$, which is constant on each interval of the form $[m\varepsilon, (m+1)\varepsilon)$ and such that its measure derivative has compact support. We shall take (remember that $\bar{u}$ is right continuous and a.e. equal to 0 out of a compact set)

$$\bar{u}^\varepsilon(x) = \bar{u}(m\varepsilon) \qquad \text{for all } x \in [x_{m-1}, x_m). \tag{2.16}$$

Notice that clearly

$$\text{Tot.Var.}(u^\varepsilon(0);\mathbb{R}) \leq \text{Tot.Var.}(u(0);\mathbb{R}) \tag{2.17}$$

and

$$\big\|\bar{u}^\varepsilon - \bar{u}\big\|_1 \leq \text{Tot.Var.}\big(u;\ \mathbb{R}\big)\varepsilon.$$

For times $t > 0$ sufficiently small, the solution $u^\varepsilon = u^\varepsilon(t,x)$ is obtained by solving the Riemann problems corresponding to the jumps of the initial approximation $\bar{u}^\varepsilon$ at the nodes x_m. By (2.2), the solutions to the Riemann problems do not overlap on the time interval $[0,\varepsilon)$, and thus $u^\varepsilon(t)$ can be prolonged up to $t = \varepsilon$.

At time $t_1 = \varepsilon$ a restarting procedure is adopted: the function $u^\varepsilon(t_1-,\cdot)$ is approximate by a new function $u^\varepsilon(t_1,\cdot)$ which is piecewise constant, having jumps exactly at the nodes $x_m = m\varepsilon$. If the total variation of the solution remains uniformly bounded in time, the approximate solution u^ε can now be constructed on the further time interval $[\varepsilon, 2\varepsilon)$, again by piecing together the solutions of the various Riemann problems determined by the jumps at the nodal points x_m. At time $t_2 = 2\varepsilon$, this solution is again approximated by a piecewise constant function, and the procedure goes on.

A key aspect of the construction is the restarting procedure. At each time $t_i = i\varepsilon$, we need to approximate $u^\varepsilon(t_i-,\cdot)$ with a piecewise constant function $u^\varepsilon(t_i,\cdot)$ having jumps precisely at the nodal points x_m. This is achieved by a random sampling technique. More precisely, we look at the number ϑ_i in the sequence $\{\vartheta_j\}_j$. On each interval $[x_{m-1}, x_m)$, the old value of our solution at the intermediate point $\vartheta_i x_m + (1-\vartheta_i)x_{m-1}$ becomes the new value over the whole interval:

$$u^\varepsilon(t_i,x) := u^\varepsilon\big(t_i-, (\vartheta_i x_m + (1-\vartheta_i)x_{m-1})\big) \qquad \text{for all } x \in [x_{m-1}, x_m).$$

We will show in Theorem 2.16 the, if the initial datum $\bar{u}$ has sufficiently small total variation, then an approximate solution can be constructed by the above algorithm for all times $t \geq 0$ and moreover

$$\text{Tot.Var.}(u^\varepsilon(t),\mathbb{R}) \leq \mathcal{O}(1)\text{Tot.Var.}(u(0),\mathbb{R}). \tag{2.18}$$

Assuming that u^ε is defined for all time $t \in [0,\infty)$, we make now the following remarks. First of all notice that u^ε restricted on the time interval $[0,T]$ is identically zero out of a compact set depending on T.

For our purposes it will be convenient to redefine u^ε inside the open strips $(i\varepsilon,(i+1)\varepsilon)\times\mathbb{R}$ as follows:

$$u^\varepsilon(t,x) := \begin{cases} u^\varepsilon((i+1)\varepsilon, m\varepsilon) & \text{if } m\varepsilon \leq x < m\varepsilon + t - i\varepsilon, \\ u^\varepsilon(i\varepsilon, m\varepsilon) & \text{if } m\varepsilon + t - i\varepsilon \leq x < (m+1)\varepsilon. \end{cases}$$

In this way, $u^\varepsilon(t,\cdot)$ becomes a compactly supported, piecewise constant function for each time $t \geq 0$ with jumps along piecewise linear curves passing through the nodes $(i\varepsilon, m\varepsilon)$.

To conclude this section, let us introduce some notations which we will be used in the next. For any grid point $(i\varepsilon, m\varepsilon)$, $i \geq 0$, $m \in \mathbb{Z}$, set

$$u^{i,m} := u^\varepsilon(i\varepsilon, m\varepsilon),$$

and assume that the Riemann problem $(u^{i,m-1}, u^{i,m})$ is solved by the collection of exact curve $\{\gamma_k^{i,m}\}_{k=1,\dots,N}$, where $\gamma_k^{i,m} = (u_k^{i,m}, v_k^{i,m}, \sigma_k^{i,m})$ is an exact curve of the k-th family with length

$s_k^{i,m}$, defined on $\mathbf{I}(s_k^{i,m})$:

$$u^{i,m} = T^N_{s_N^{i,m}} \circ \cdots \circ T^1_{s_1^{i,m}} u^{i,m-1}.$$

Define also

$$V_k^+(t) := \sum_{m\in\mathbb{Z}} \big[s_k^{i,m}\big]^+, \qquad V_k^-(t) := -\sum_{m\in\mathbb{Z}} \big[s_k^{i,m}\big]^-, \quad \text{if } t \in [i\varepsilon, (i+1)\varepsilon). \tag{2.19}$$

Let us introduce also the following notation for the transversal and cubic amounts of interaction and for the amount of cancellation related to the two Riemann problems $(u_{i,m-1}, u_{i-1,m-1})$, $(u_{i-1,m-1}, u_{i,m})$ which interact at grid point $(i\varepsilon, (m-1)\varepsilon)$:

$$\mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) := \mathtt{A}^{\text{trans}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}),$$

and for $k = 1, \dots, n$,

$$\mathtt{A}_k^{\text{cubic}}(i\varepsilon, m\varepsilon) := \mathtt{A}_k^{\text{cubic}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}),$$

$$\mathtt{A}_k^{\text{canc}}(i\varepsilon, m\varepsilon) := \mathtt{A}_k^{\text{canc}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}),$$

$$\mathtt{A}_k^{\text{cr}}(i\varepsilon, m\varepsilon) := \mathtt{A}_k^{\text{cr}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}).$$

2.4. Known Lyapunov functionals

In this section we recall the definitions and the basic properties of three already well known functionals, namely the *total variation functional*, the functional introduced by Glimm in [Gli65] which controls the transversal amounts of interaction and the functional introduced by Bianchini in [Bia03], which controls the cubic amounts of interaction.

Let $\varepsilon > 0$ be fixed and let u^ε be the corresponding Glimm approximate solution. The following definitions hold, for the moment, on the largest time interval on which u^ε can be defined. However, Theorem 2.16 below shows that u^ε can be defined on the whole half-plane $[0, \infty) \times \mathbb{R}$.

Definition 2.11. Define the *total variation along curves* as

$$V(t) := \sum_{k=1}^N \sum_{m\in\mathbb{Z}} |s_k^{i,m}|, \qquad \text{for any } t \in [i\varepsilon, (i+1)\varepsilon).$$

Define the *transversal interaction functional* as

$$Q^{\text{trans}}(t) := \sum_{k=1}^N \sum_{h=1}^{k-1} \sum_{m>m'} |s_k^{i,m'}||s_h^{i,m}|, \qquad \text{for any } t \in [i\varepsilon, (i+1)\varepsilon).$$

Define the *cubic interaction functional* as

$$Q^{\text{cubic}}(t) := \sum_{k=1}^N \sum_{m,m'\in\mathbb{Z}} \int_{\mathbf{I}(s_k^{i,m})} \int_{\mathbf{I}(s_k^{i,m'})} \big|\sigma_k^{i,m}(\tau) - \sigma_k^{i,m'}(\tau')\big| d\tau' d\tau.$$

Notice the all the objects just introduced depend on ε, even if we do not write this dependence explicitly.

Remark 2.12. The three functionals $t \mapsto V(t), Q^{\text{trans}}(t), Q^{\text{cubic}}(t)$ are local in time, i.e. their value at time $\bar t$ depends only on the solution $u^\varepsilon(\bar t)$ at time $\bar t$ and not on the solution at any other time $t \neq \bar t$. On the contrary, the functionals $\mathfrak{Q}_k, k = 1, \dots, N$ we will introduce in Section 3.5 to bound the difference in speed of the wavefronts before and after the interactions are non-local in time, i.e. their definition requires the knowledge of the whole solution in $[0, \infty) \times \mathbb{R}$.

REMARK 2.13. The functional Q^{cubic} has been introduced first in [BB02]. The first work where the idea of multiplying the product of the strengths by a factor which takes into account the relative speeds is the paper [Liu81].

The following statements hold: for the proofs, see [Bre00], [Bia03].

PROPOSITION 2.14. *There exists a constant $C > 0$, depending only of the flux f, such that for any time $t \geq 0$*

$$\frac{1}{C}\mathrm{Tot.Var.}(u(t)) \leq V(t) \leq C\mathrm{Tot.Var.}(u(t)).$$

THEOREM 2.15. *The following hold:*

(1) *the functionals $t \mapsto V(t), Q^{\mathrm{trans}}(t), Q^{\mathrm{cubic}}(t)$ are constant on each interval $[i\varepsilon, (i+1)\varepsilon)$;*
(2) *they are bounded by powers of the* Tot.Var.$(u(t))$ *as follows:*

$$\begin{aligned} V(t) &\leq C\mathrm{Tot.Var.}(u^\varepsilon(t)), \\ Q^{\mathrm{trans}}(t) &\leq C\mathrm{Tot.Var.}(u^\varepsilon(t))^2, \\ Q^{\mathrm{cubic}}(t) &\leq C\mathrm{Tot.Var.}(u^\varepsilon(t))^3; \end{aligned}$$

(3) *there exist constants $c_1, c_2, c_3 > 0$, depending only on the flux f, such that for any $i \in \mathbb{N}$, defining*

$$Q^{\mathrm{known}}(t) := c_1 V(t) + c_2 Q^{\mathrm{trans}}(t) + c_3 Q^{\mathrm{cubic}}(t),$$

it holds

$$\sum_{m\in\mathbb{Z}} \left[\mathtt{A}^{\mathrm{trans}}(i\varepsilon, m\varepsilon) + \sum_{k=1}^{N} \Big(\mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, m\varepsilon) + \mathtt{A}_k^{\mathrm{cubic}}(i\varepsilon, m\varepsilon) \Big) \right] \leq Q^{\mathrm{known}}((i-1)\varepsilon) - Q^{\mathrm{known}}(i\varepsilon).$$

Using the previous proposition and theorem, we now prove that it is possible to construct a Glimm approximate solution u^ε for any $\varepsilon > 0$ provided that Tot.Var.$(\bar{u}; \mathbb{R}) \ll 1$. The proof is a standard technique in Hyperbolic Conservation Laws. We explicitly prove the theorem for the sake of completeness.

THEOREM 2.16. *If* Tot.Var.$(\bar{u}; \mathbb{R})$ *is sufficiently small, then for every $\varepsilon > 0$ it is possible to construct a Glimm approximate solution u^ε (with the algorithm described in Section 2.3) defined for all times $t \in [0, \infty)$.*

PROOF. Fix $\delta_2 > 0$. By Proposition 2.1 there exist $\delta_1 > 0$ and $\delta_0 > 0$ such that

(1) every Riemann problem (u^L, u^R) with $u^L, u^R \in B(0, \delta_1)$ admits a unique, self-similar, right continuous, vanishing viscosity solution, taking values in $B(0, \delta_2)$;
(2) it holds

$$\frac{C^2}{c_1}\delta_0\Big(c_1 + c_2\delta_0 + c_3\delta_0^2\Big) \leq \delta_1,$$

where C, c_1, c_2, c_3 are the constant introduced in the statements of Proposition 2.14 and Theorem 2.15;
(3) every Riemann problem (u^L, u^R) with $u^L, u^R \in B(0, \delta_0)$ admits a unique, self-similar, right continuous, vanishing viscosity solution, taking values in $B(0, \delta_1)$.

Assume that Tot.Var.$(\bar{u}; \mathbb{R}) \leq \delta_0$. We prove now, by induction on $i \in \mathbb{N}$, that it is possible to define u^ε on the time interval $[0, i\varepsilon]$ with $u^\varepsilon(i\varepsilon, \cdot)$ taking values in $B(0, \delta_1)$. For $i = 0$ we choose $u^\varepsilon(0, x) = \bar{u}^\varepsilon(x)$, where $\bar{u}^\varepsilon$ is the approximation of $\bar{u}$ defined in (2.16). Assume now that the solution is defined on $[0, (i-1)\varepsilon]$ with $u^\varepsilon((i-1)\varepsilon, \cdot)$ taking values in $B(0, \delta_1)$ and let us prove that we can prolong it up to time $i\varepsilon$ with $u^\varepsilon(i\varepsilon,\cdot)$ taking values in $B(0,\delta_1)$. Clearly, since $u^\varepsilon((i-1)\varepsilon)$ takes values in $B(0,\delta_1)$, we can prolong u^ε up to time $i\varepsilon$ with values in $B(0,\delta_2)$. Now using Proposition 2.14 and Theorem 2.15 we get

$$\begin{aligned}
\text{Tot.Var.}\Big(u^\varepsilon(i\varepsilon-,\cdot);\mathbb{R}\Big) &\le CV(i\varepsilon)\\
&\le \frac{C}{c_1}\Big[c_1V(i\varepsilon)+c_2Q^{\text{trans}}(i\varepsilon)+c_3Q^{\text{cubic}}(i\varepsilon)\Big]\\
&\le \frac{C}{c_1}\Big[c_1V(0)+c_2Q^{\text{trans}}(0)+c_3Q^{\text{cubic}}(0)\Big]\\
&\le \frac{C^2}{c_1}\Big[c_1\text{Tot.Var.}(\bar u^\varepsilon;\mathbb{R})+c_2\text{Tot.Var.}(\bar u^\varepsilon;\mathbb{R})^2+c_3\text{Tot.Var.}(\bar u^\varepsilon;\mathbb{R})^3\Big]\\
&\le \frac{C^2}{c_1}\delta_0\Big[c_1+c_2\delta_0+c_3\delta_0^2\Big]\\
\text{(by Point (2) above)}\quad &\le \delta_1,
\end{aligned}$$

thus proving that $u^\varepsilon(i\varepsilon)$ takes values in $B(0,\delta_1)$ and concluding the proof of the theorem. □

CHAPTER 3

A quadratic interaction estimate

In this chapter we prove the first result of this thesis, namely Theorem A in the Introduction, which is the natural extension to a Glimm approximate solution of the Cauchy problem

$$\begin{cases} u_t + F(u)_x = 0, \\ u(t=0)\bar{u}, \end{cases} \tag{3.1}$$

of the quadratic interaction estimate (18) presented in the Introduction, where the reader can find an extensive discussion on the history and the interest of this estimate. Theorem A is the final outcome of papers [BM14a], [BM14b], [BM15b]. In particular we will present here the result in the most general setting, namely the one considered in [BM15b]. Before entering into technical details, we recall now the statement of Theorem A.

Let (u^L, u^M), (u^M, u^R) be two Riemann problems with a common state u^M, and consider the Riemann problem (u^L, u^R). We have shown in Section 2.1 that if $|u^M - u^L|, |u^R - u^M| \ll 1$, then one can solve the three Riemann problems as follows:

$$u^M = T^N_{s'_N} \circ \cdots \circ T^1_{s'_1} u^L, \qquad u^R = T^N_{s''_N} \circ \cdots \circ T^1_{s''_1} u^M, \qquad u^R = T^N_{s_N} \circ \cdots \circ T^1_{s_1} u^L,$$

where for each $k = 1, \dots, N$, $s'_k, s''_k, s_k \in \mathbb{R}$ and $(s, u) \mapsto T^k_s u$ is the map which at each left state u associates the right state $T^k_s u$ such that the Riemann problem $(u, T^k_s u)$ has an entropy admissible solution made only by wavefronts with total strength $|s|$ belonging to the k-th family.

Writing for brevity (formula (1.1))

$$\mathbf{I}(s) = \big[\min\{s, 0\}, \max\{s, 0\}\big] \setminus \{0\},$$

let us denote by

$\sigma'_k : \mathbf{I}(s_k) \to (\hat{\lambda}_{k-1}, \hat{\lambda}_k)$	the speed function of the wavefronts of the k-th family for the Riemann problem (u^L, u^M),
$\sigma''_k : s'_k + \mathbf{I}(s''_k) \to (\hat{\lambda}_{k-1}, \hat{\lambda}_k)$	the speed function of the wavefronts of the k-th family for the Riemann problem (u^M, u^R),
$\sigma_k : \mathbf{I}(s_k) \to (\hat{\lambda}_{k-1}, \hat{\lambda}_k)$	the speed function of the wavefronts of the k-th family for the Riemann problem (u^L, u^R).

See (2.2) for the definition of $\hat{\lambda}_h$, $h = 0, 1, \dots, N$. We assume that σ''_k is defined on $s'_k + \mathbf{I}(s''_k)$ instead of $\mathbf{I}(s''_k)$). Let us consider the L^1-norm of the speed difference between the waves of the Riemann problems (u_L, u_M), (u_M, u_R) and the outgoing waves of (u_L, u_R):

$$\Delta\sigma_k(u^L, u^M, u^R) := \begin{cases} \big\| (\sigma'_k \cup \sigma''_k) - \sigma_k \big\|_{L^1(\mathbf{I}(s'_k + s''_k) \cap \mathbf{I}(s_k))} & \text{if } s'_k s''_k \geq 0, \\ \big\| (\sigma'_k \vartriangle \sigma''_k) - \sigma_k \big\|_{L^1(\mathbf{I}(s'_k + s''_k) \cap \mathbf{I}(s_k))} & \text{if } s'_k s''_k < 0, \end{cases}$$

where $\sigma'_k \cup \sigma''_k$ is the function obtained by piecing together σ'_k, σ''_k, while $\sigma'_k \vartriangle \sigma''_k$ is the restriction of σ'_k to $\mathbf{I}(s'_k + s''_k)$ if $|s'_k| \geq |s''_k|$ or $\sigma''_k \llcorner_{\mathbf{I}(s'_k+s''_k)}$ in the other case, see formulas (1.2), (1.3).

Now consider a right continuous ε-approximate solution constructed by the Glimm scheme (see Section 2.3); by simplicity, for any grid point $(i\varepsilon, m\varepsilon)$ denote by

$$\Delta\sigma_k(i\varepsilon, m\varepsilon) := \Delta\sigma_k(u^{i,m-1}, u^{i-1,m-1}, u^{i,m})$$

the change in speed of the k-th wavefronts at the grid point $(i\varepsilon, m\varepsilon)$ arriving from points $(i\varepsilon, (m-1)\varepsilon)$, $((i-1)\varepsilon, (m-1)\varepsilon)$, where $u^{j,r} := u(j\varepsilon, r\varepsilon)$. We can now state the main result of this chapter, namely Theorem A in the Introduction, which says that the sum over all grid points of the change in speed is bounded by a quantity which depends only on the flux f and the total variation of the initial datum and does not depend on ε.

Theorem A. *It holds*

$$\sum_{i=1}^{+\infty} \sum_{m\in\mathbb{Z}} \Delta\sigma_k(i\varepsilon, m\varepsilon) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R})^2, \tag{3.2}$$

where $\mathcal{O}(1)$ is a quantity which depends only on the flux f.

We explicitly notice that $\Delta\sigma_k$ is the *variation of the speed of the waves when joining two Riemann problems.*

The proof of Theorem A follows a classical approach used in hyperbolic system of conservation laws in one space dimension.

We first prove a *local* estimate. For the couple of Riemann problems (u^L, u^M), (u^M, u^R), we define the quantity

$$\begin{aligned}\mathtt{A}(u^L, u^M, u^R) := {}& \mathtt{A}^{\text{trans}}(u^L, u^M, u^R) \\ & + \sum_{h=1}^{N} \Big(\mathtt{A}_h^{\text{quadr}}(u^L, u^M, u^R) + \mathtt{A}_h^{\text{canc}}(u^L, u^M, u^R) + \mathtt{A}_h^{\text{cubic}}(u^L, u^M, u^R) \Big),\end{aligned} \tag{3.3}$$

which we will call the *global amount of interaction* of the two merging RPs (u^L, u^M), (u^M, u^R). Three of the terms in the r.h.s. of (3.3) have already been defined in 2.2, namely

$\mathtt{A}^{\text{trans}}(u^L, u^M, u^R)$ is the *transversal amount of interaction* (see Definition 2.5);

$\mathtt{A}_h^{\text{canc}}(u^L, u^M, u^R)$ is the *amount of cancellation* of the h-th family (see Definition 2.8);

$\mathtt{A}_h^{\text{cubic}}(u^L, u^M, u^R)$ is the *cubic amount of interaction* of the h-th family (see Definition 2.6).

The term $\mathtt{A}_h^{\text{quadr}}(u^L, u^M, u^R)$, which we will call the *quadratic amount of interaction* of the h-th family, will be defined in Definition 3.16 and was introduced for the first time in [BM15b]. The local estimate we will prove is the following: for all $k = 1, \dots, N$,

$$\Delta\sigma_k(u^L, u^M, u^R) \leq \mathcal{O}(1)\mathtt{A}(u^L, u^M, u^R). \tag{3.4}$$

This is done in Section 3.2, Theorem 3.18.

Next we show a *global* estimate, based on a new interaction potential. For any grid point $(i\varepsilon, m\varepsilon)$ define

$$\mathtt{A}(i\varepsilon, m\varepsilon) := \mathtt{A}(u^{i,m-1}, u^{i-1,m-1}, u^{i,m})$$

as the amount of interaction at the grid point $(i\varepsilon, m\varepsilon)$, and similarly let $\mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon)$, $\mathtt{A}_h^{\text{canc}}(i\varepsilon, m\varepsilon)$, $\mathtt{A}_h^{\text{cubic}}(i\varepsilon, m\varepsilon)$, $\mathtt{A}_h^{\text{quadr}}(i\varepsilon, m\varepsilon)$ be the transversal amount of interaction, amount of cancellation, cubic amount of interaction at the grid point $(i\varepsilon, m\varepsilon)$, respectively.

For every fixed time $T > 0$, we will introduce a new interaction potential $\Upsilon : [0,T] \to [0,\infty)$ with the following properties:

(1) it is uniformly bounded at time $t = 0$: in fact,

$$\Upsilon(0) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R})^2;$$

(2) it is constant on time intervals $[(i-1)\varepsilon, i\varepsilon)$;

(3) at any time $i\varepsilon \in [0,T)$, it decreases at least of $\frac{1}{2}\sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon)$.

It is fairly easy to see that Points (1), (2), (3) above, together with inequality (3.4), imply that

$$\sum_{i=1}^{I} \sum_{m\in\mathbb{Z}} \Delta\sigma_k(i\varepsilon, m\varepsilon) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R})^2,$$

where $I := \max\{i \in \mathbb{N} \mid i\varepsilon \in [0,T)\}$. As a consequence, by the arbitrarity of T and by Point (1) above, we get Theorem A.

The potential Υ is constructed as follows. We define a positive functional $\mathfrak{Q}(t) : [0,T] \to [0,\infty)$, bounded by $\mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R})^2$ at $t = 0$, which satisfies the following inequality (see Theorem 3.58):

$$\begin{aligned}
\mathfrak{Q}(i\varepsilon) - \mathfrak{Q}((i-1)\varepsilon) \leq{}& -\sum_{m\in\mathbb{Z}}\sum_{h=1}^{N} \mathtt{A}_h^{\text{quadr}}(i\varepsilon, m\varepsilon) + \mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R}) \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \\
={}& -\big(1 - \mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R})\big) \sum_{m\in\mathbb{Z}}\sum_{h=1}^{N} \mathtt{A}_h^{\text{quadr}} i\varepsilon, m\varepsilon) \\
&+ \mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R}) \sum_{m\in\mathbb{Z}} \mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) \\
&+ \mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R}) \sum_{m\in\mathbb{Z}}\sum_{h=1}^{N} \Big(\mathtt{A}_h^{\text{canc}}(i\varepsilon, m\varepsilon) + \mathtt{A}_h^{\text{cubic}}(i\varepsilon, m\varepsilon)\Big).
\end{aligned} \tag{3.5}$$

We have already seen in Section 2.4, Theorem 2.15 that there exists a uniformly bounded, decreasing potential $Q^{\text{known}}(t)$ such that at each time $i\varepsilon$

$$\sum_{m\in\mathbb{Z}} \Bigg[\mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) + \sum_{h=1}^{N} \Big(\mathtt{A}_h^{\text{canc}}(i\varepsilon, m\varepsilon) + \mathtt{A}_h^{\text{cubic}}(i\varepsilon, m\varepsilon)\Big)\Bigg] \leq Q^{\text{known}}((i-1)\varepsilon) - Q^{\text{known}}(i\varepsilon). \tag{3.6}$$

Hence, it is straightforward to see from (3.5), (3.6) that we can find a constant M big enough, such that the potential

$$\Upsilon(t) := \mathfrak{Q}(t) + MQ^{\text{known}}(t)$$

satisfies Properties (1)-(3) above, provided that $\text{Tot.Var.}(\bar{u}; \mathbb{R}) \ll 1$.

Notice that the functionals Υ and $\mathfrak{Q}$ depends on the parameter $\varepsilon >$ and on the fixed time T, even if we do not explicitly write this dependence.

REMARK 3.1. In the paper [Liu77], for GNL or LD systems, the author proves an estimate, analogous to (3.2), on the change in speed of a suitable partition of the elementary waves of the Riemann problem at each grid point. This discrete partition is needed in order to group rarefaction waves into finitely many packets, which are then traced in time.
A first extension of this idea is in [AM11b], where due to the generic flux F one has also to partition the shock waves.

The construction presented in this paper is thus different in two main aspects. First, our partition is *continuous*, i.e. the variable w, indexing the wave, varies in a subset of the real numbers, while in all the previous cases it is contained in a discrete set.
Secondly, the quadratic estimate of [**Liu77**, Lemma 3.2-(iii)] is much easier due to the decrease of the original Glimm functional (which is *quadratic* w.r.t. the total variation), while in our case one of the main points is precisely the construction of a quadratic decreasing functional.

Notice also that without Theorem A, one cannot show the convergence of the Glimm scheme to the entropic solution when the sampling sequence ϑ_i, $i \in \mathbb{N}$, is equidistributed.

REMARK 3.2. The definition of the functional $\mathfrak{Q}$ we provide in this chapter is not exactly the one proposed in [**BM15b**]. Indeed, as we have already pointed out in the Introduction, the functional proposed in [**BM15b**], which works perfectly to prove Theorem A, is not sharp enough to prove the estimate on the convergence rate of the Glimm scheme, namely Theorem B which will be proved in Chapter 4. We decided therefore to introduce already in this chapter this stronger definition of $\mathfrak{Q}$ which is suitable to prove both Theorem A and Theorem B.

REMARK 3.3. The functional $\mathfrak{Q}$, which is defined on the time interval $[0,T]$, where T is an arbitrary number, could be defined on the whole half line $[0,+\infty)$. However, this would require some technical trick which can be found in the cited paper [**BM15b**], but which we decided to avoid here, for the sake of simplicity.

Structure of the chapter. The Chapter is organized as follows.

In Section 3.1 we present some basic interaction estimates on the exact curves (see Section 2.1) used to find the vanishing viscosity solution of the Riemann problem. Some of these estimates will be used in this chapter, while some others will be used in the next chapters. We collect all them here for the sake of convenience.

Section 3.2 devoted to prove the *local* part of the proof of Theorem A, as explained before. In particular we will consider two contiguous Riemann problems (u^L, u^M), (u^M, u^R) which are merging, producing the Riemann problem (u^L, u^R) and we will introduce a *global amount of interaction* $\mathtt{A}$, which bounds the L^1-distance between the speed of the wavefronts before and after the interaction, i.e. the σ-component of the elementary curves.

In Section 3.3 we define the notion of *wave tracing* of an approximate solution u^ε to the Cauchy problem (3.1), obtained by the Glimm scheme, together with some additional objects, and we explicitly construct a wave tracing satisfying some useful further properties.

Starting with Section 3.4 we enter in the heart of our construction. We introduce in fact the notion of *pair of waves* (w,w') *which have already interacted* and *pair of waves* (w,w') *which have never interacted* at time $\bar{t}$. For any pair of waves (w,w') and for any fixed times $t_1 \leq t_2$, we define an interval of waves $\mathcal{I}(t_1,t_2,w,w')$ and a partition $\mathcal{P}(t_1,t_2,w,w')$ of this interval: these objects in some sense summarize the past "common" history of the two waves, from the time of last splitting before t_1 (or from the last time in which one of them is created) up to the time t_2.

Now we have all the tools we need to define the functional $\mathfrak{Q}_k$ for $k = 1,\dots,n$ and to prove that it satisfies the inequality (3.5), thus obtaining the *global* part of the proof of Theorem A. This is done in the final Section 3.5.

3.1. Basic interaction estimates

In this section we prove some basic estimates which will be used in the following to analyze the interaction between two or more merging Riemann problems. In particular the results we are going to present now will be widely used in Sections 3.2, 4.3 and 5.3. All the proofs are based on the results presented in Section 1.2 about convex functions and on the fact that the curves $\Gamma_k(u^L, s)$ are obtained as fixed point of a contraction. In what follows we always assume that the length of all the curves we consider is "small enough".

We start with the description of a general situation we will encounter many times. Let $k = 1, \dots, N$ be a fixed family. Let $\{\gamma^p\}_{p=1,\dots,P}$ be a collection of P curves,

$$\gamma^p = (u^p, v^p, \sigma^p) : I^p \to \mathcal{D}_{k(p)} \subseteq \mathbb{R}^{n+2}$$

with I^p an interval of the form $a + \mathbf{I}(s^p)$ for some $a \in \mathbb{R}$, u^p, v^p of class $C^{1,1}$, σ^p Lipschitz continuous and $k(p)$ an index in $\{1, \dots, N\}$. The set $\mathcal{D}_k$ was defined in (2.4). Notice that here we do not require that γ^p is an exact curve. Denote by f^p the reduced flux associated to γ^p, i.e. any $C^{1,1}$ map which satisfies

$$Df^p(\tau) = \tilde{\lambda}_{k(p)}\Big(u^p(\tau), v^p(\tau), \sigma^p(\tau)\Big).$$

Definition 3.4. We say that $\gamma^1, \dots, \gamma^P$ *satisfies the assumption* ($\star$) if, setting

$$a^p := \sum_{p' \leq p} s^p \text{ for any } p = 0, \dots, P,$$

it holds

$$\begin{aligned} &\text{for any } p = 1, \dots, P \\ &k(p) = k \text{ for some fixed } k \in \{1, \dots, N\} \text{ independent of } p, \\ &\gamma^p \text{ and } f^p \text{ are defined on } a^{p-1} + \mathbf{I}(s^p) \\ &\text{and } f^p(a^{p-1}) = f^{p-1}(a^{p-1}) \end{aligned} \tag{$\star$}$$

Notice that the condition on f can be always be assumed, because the reduced fluxes are defined up to an additive constant.

Definition 3.5. We say that $(\gamma^1, \dots, \gamma^P)$ is a *collection of consecutive curves* if $u^p(a^{p-1}) = u^{p-1}(a^{p-1})$.

Recall that a generic curve $\mathbf{I}(s) \to \mathcal{D}_k \subseteq \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ is denoted by $\gamma_k = (u_k, v_k, \sigma_k)$, where the index is just to remember that γ_k takes values in $\mathcal{D}_k$.

The following lemma is just an observation on the second derivative of the reduced flux.

Lemma 3.6. *Let $\gamma_k = (u_k, v_k, \sigma_k) := \Gamma_k(u_0, s)$ be the exact curve of length s starting in u_0 and let f_k be the reduced flux associated to γ_k i.e.*

$$f_k(\tau) := \int_0^\tau \tilde{\lambda}_k(u_k(\varsigma), v_k(\varsigma), \sigma_k(\varsigma))d\varsigma.$$

Then, for a.e. $\tau \in \boldsymbol{I}(s)$, it holds

$$\frac{d^2 f_k}{d\tau^2}(\tau) = \frac{\partial \tilde{\lambda}_k}{\partial u}\big(\gamma_k(\tau)\big)\tilde{r}_k\big(\gamma_k(\tau)\big) + \frac{\partial \tilde{\lambda}_k}{\partial v_k}\big(\gamma_k(\tau)\big)\bigg[\tilde{\lambda}_k\big(\gamma_k(\tau)\big) - \frac{d \operatorname{conv}_{[0,s]} f_k}{d\tau}(\tau)\bigg]. \tag{3.7}$$

Proof. Observe that

$$\frac{\partial \tilde{\lambda}_k}{\partial \sigma_k}\big(\gamma_k(\tau)\big)\frac{d\sigma_k}{d\tau}(\tau) = 0 \qquad \text{for } \mathcal{L}^1\text{-a.e. } \tau \in [0, s].$$

Namely, if $\frac{d\sigma_k(\tau)}{d\tau} \neq 0$ for some τ, then $v_k(\tau) = 0$ and thus, by (2.6),

$$\left|\frac{\partial \tilde{\lambda}_k}{\partial \sigma_k}\big(\gamma_k(\tau)\big)\right| \leq \mathcal{O}(1)\big|v_k(\tau)\big| = 0.$$

As a consequence, formula (3.7) holds for a.e. $\tau \in \mathbf{I}(s)$. □

LEMMA 3.7 (Translation of the starting point). *Let $\gamma_k = \Gamma_k(u_0, s) = (u_k, v_k, \sigma_k)$ and $\gamma'_k = \Gamma_k(u'_0, s) = (u'_k, v'_k, \sigma'_k)$. Denote by f_k, f'_k the reduced flux associated to γ_k, γ'_k respectively. Then it holds*

$$\|u_k - u'_k\|_\infty \leq \big(1 + \mathcal{O}(1)|s|\big)|u_0 - u'_0|, \qquad \|v_k - v'_k\|_\infty \leq \mathcal{O}(1)|s||u_0 - u'_0|,$$

$$\|\sigma_k - \sigma'_k\|_1 \leq \mathcal{O}(1)|s||u_0 - u'_0|, \qquad \left\|\frac{d^2 f_k}{d\tau^2} - \frac{d^2 f'_k}{d\tau^2}\right\|_1 \leq \mathcal{O}(1)|s||u_0 - u'_0|.$$

PROOF. It holds

$$\begin{cases} u_k(\tau) = u_0 + \displaystyle\int_0^\tau \tilde r_k(\gamma_k(\varsigma))d\varsigma \\ v_k(\tau) = f_k(\tau) - \operatorname*{conv}_{[0,s]} f_k(\tau) \\ \sigma_k(\tau) = \dfrac{d}{d\tau}\operatorname*{conv}_{[0,s]} f_k(\tau) \end{cases} \qquad \begin{cases} u'_k(\tau) = u'_0 + \displaystyle\int_0^\tau \tilde r_k(\gamma'_k(\varsigma))d\varsigma \\ v'_k(\tau) = f'_k(\tau) - \operatorname*{conv}_{[0,s]} f'_k(\tau) \\ \sigma'_k(\tau) = \dfrac{d}{d\tau}\operatorname*{conv}_{[0,s]} f'_k(\tau) \end{cases}$$

Consider the curve $\widetilde\gamma_k(\tau) = \gamma_k(\tau) + (u'_0 - u_0, 0, 0)$, i.e. the translation of γ_k from the starting point $(u_0, 0, \sigma_k(0))$ to $(u'_0, 0, \sigma_k(0))$ and set $\widetilde{\widetilde\gamma}_k := \mathcal{T}(\widetilde\gamma_k)$. Denote by $\tilde f_k$ the reduced flux associated to $\widetilde\gamma_k$. The curves $\widetilde\gamma_k$ and $\widetilde{\widetilde\gamma}_k$ satisfy the following systems

$$\begin{cases} \widetilde u_k(\tau) = u'_0 + \displaystyle\int_0^\tau \tilde r_k(\gamma_k(\varsigma))d\varsigma \\ \widetilde v_k(\tau) = f'_k(\tau) - \operatorname*{conv}_{[0,s]} f_k(\tau) \\ \widetilde\sigma_k(\tau) = \dfrac{d}{d\tau}\operatorname*{conv}_{[0,s]} f_k(\tau) \end{cases} \qquad \begin{cases} \widetilde{\widetilde u}_k(\tau) = u'_0 + \displaystyle\int_0^\tau \tilde r_k(\widetilde\gamma_k(\varsigma))d\varsigma \\ \widetilde{\widetilde v}_k(\tau) = \tilde f_k(\tau) - \operatorname*{conv}_{[0,s]} \tilde f_k(\tau) \\ \widetilde{\widetilde\sigma}_k(\tau) = \dfrac{d}{d\tau}\operatorname*{conv}_{[0,s]} \tilde f_k(\tau) \end{cases}$$

Let us prove now the first inequality. Using the Contraction Mapping Principle, we get

$$\|u_k - u'_k\|_\infty \leq \|u_k - \tilde u_k\|_\infty + \|\tilde u_k - u'_k\|_\infty \leq |u_0 - u'_0| + D(\widetilde\gamma_k, \gamma'_k) \leq |u_0 - u'_0| + 2D(\widetilde\gamma_k, \mathcal{T}(\widetilde\gamma_k)),$$

being the map $\mathcal{T}$ a contraction with constant $1/2$.
Since $\widetilde\gamma_k$ is obtained from γ_k by translation of the initial point of the u-component, we get

$$\begin{aligned} |\tilde{\tilde u}_k(\tau) - \tilde u_k(\tau)| &\leq \int_0^\tau \big|\tilde r_k(\widetilde\gamma_k(\varsigma)) - \tilde r_k(\gamma_k(\varsigma))\big|d\varsigma \\ &\leq \int_0^\tau \left(\left\|\frac{\partial \tilde r_k}{\partial u}\right\|_\infty \big|\tilde u_k(\varsigma) - u_k(\varsigma)\big| + \left\|\frac{\partial \tilde r_k}{\partial v_k}\right\|_\infty \big|\tilde v_k(\varsigma) - v_k(\varsigma)\big| + \left\|\frac{\partial \tilde r_k}{\partial \sigma_k}\right\|_\infty \big|\tilde\sigma_k(\varsigma) - \sigma_k(\varsigma)\big|\right) d\varsigma \\ &= \int_0^\tau \left\|\frac{\partial \tilde r_k}{\partial u}\right\|_\infty \big|\tilde u_k(\varsigma) - u_k(\varsigma)\big| d\varsigma \\ &\leq \mathcal{O}(1)|u_0 - u'_0||s|. \end{aligned}$$

Similarly,

$$\big\|\widetilde{\widetilde v}_k - \widetilde v_k\big\|_\infty \leq \mathcal{O}(1)|s||u_0 - u'_0|, \quad \big\|\widetilde{\widetilde\sigma}_k - \widetilde\sigma_k\big\|_1 \leq \mathcal{O}(1)|s||u_0 - u'_0|,$$

and thus

$$D(\widetilde\gamma_k, \widetilde{\widetilde\gamma}_k) \leq \mathcal{O}(1)|s||u_0 - u'_0|. \tag{3.9}$$

Hence

$$\|u_k - u'_k\|_\infty \leq \big(1 + \mathcal{O}(1)|s|\big)|u_0 - u'_0|.$$

In a similar way

$$\|v_k - v'_k\|_\infty \leq \|v_k - \widetilde v_k\|_\infty + \|\widetilde v_k - v'_k\|_\infty \leq \|\widetilde v_k - v'_k\|_\infty \leq D(\widetilde\gamma_k, \gamma'_k) \leq 2D\big(\widetilde\gamma_k, \mathcal{T}(\widetilde\gamma_k)\big),$$

and

$$\|\sigma_k - \sigma'_k\|_1 \le \|\sigma_k - \widetilde{\sigma}_k\|_1 + \|\widetilde{\sigma}_k - \sigma'_k\|_1 \le \|\widetilde{\sigma}_k - \sigma'_k\|_1 \le D(\widetilde{\gamma}_k, \gamma'_k) \le 2D\big(\widetilde{\gamma}_k, \mathcal{T}(\widetilde{\gamma}_k)\big).$$

A further application of (3.9) yields the estimates on v_k, σ_k.

Finally, using the chain rule, Lemma 3.6, Proposition 1.11 and the first part of the proof, we get

$$\begin{aligned}\left\|\frac{d^2 f_k}{d\tau^2} - \frac{d^2 f'_k}{d\tau^2}\right\|_1 &\le \mathcal{O}(1)\left[\int_0^s \Big(|u_k(\tau) - u'_k(\tau)| + |v_k(\tau) - v'_k(\tau)| + |\sigma_k(\tau) - \sigma'_k(\tau)|\Big) d\tau\right] \\ &\le \mathcal{O}(1)|s||u_0 - u'_0|.\end{aligned} \tag{3.10}$$

This concludes the proof. □

LEMMA 3.8 (Translation of many curves). *Let k be a fixed family. Let $\gamma_k^1, \dots, \gamma_k^P$ be a collection of P exact curves of the k-th family satisfying the assumption ($\star$). Let $u^L \in \mathbb{R}$ be a fixed state. Define by recursion the collection of P consecutive curves of the k-th family*

$$\hat{\gamma}_k^1 = (\hat{u}_k^1, \hat{v}_k^1, \hat{\sigma}_k^1) := \Gamma_k\big(u^L, s_k^1\big), \qquad \hat{\gamma}_k^p = (\hat{u}_e^p, \hat{v}_k^p, \hat{\sigma}_k^p) := \Gamma_k\big(u_k^{p-1}(a_k^{p-1}), s_k^p\big),$$

where $a_k^p := \sum_{h \le k} s_h^p$. Assume that also $\hat{\gamma}_k^1, \dots, \hat{\gamma}_k^P$ satisfy the assumption ($\star$). Then for any $p = 1, \dots, P$,

$$\|u_k^p - \hat{u}_k^p\|_\infty \le \mathcal{O}(1) \sum_{q=1}^p \big|u_k^q(a_k^{q-1}) - u_k^{q-1}(a_k^{q-1})\big|$$

and

$$\|v_k^p - \hat{v}_k^p\|_\infty,\ \|\sigma_k^p - \hat{\sigma}_k^p\|_1 \le \mathcal{O}(1)|s_k^p| \sum_{q=1}^p \big|u_k^q(a_k^{q-1}) - u_k^{q-1}(a_k^{q-1})\big|.$$

PROOF. We first prove by induction on p that there exists a constant $C > 0$ depending only on f such that

$$\|u_k^p - \hat{u}_k^p\|_\infty \le Ce^{C\sum_{i=1}^p |s_k^i|} \sum_{q=1}^p \big|u^q(a_k^{q-1}) - u_k^{q-1}(a_k^{q-1})\big| \tag{3.11}$$

for any $p = 1, \dots, P$, where $u_k^0(a_k^0) := u^L$. For $p = 1$, (3.11) is an immediate consequence of Lemma 3.7 and the fact that $1 + C|s_k^1| \le e^{s_k^1}$. Now assume that (3.11) is proved for p and let us prove it for $p+1$. Again by Lemma 3.7, we have that

$$\begin{aligned}\big\|u_k^{p+1} - \hat{u}_k^{p+1}\big\|_\infty &\le \big(1 + C|s_k^{p+1}|\big)\big|u_k^{p+1}(a_k^p) - \hat{u}_k^{p+1}(a_k^p)\big| \\ &\le e^{C|s_k^{p+1}|}\big|u_k^{p+1}(a_k^p) - \hat{u}_k^p(a_k^p)\big| \\ &\le e^{C|s_k^{p+1}|}\Big(\big|u_k^{p+1}(a_k^p) - u_k^p(a_k^p)\big| + \big|u_k^p(a_k^p) - \hat{u}_k^p(a_k^p)\big|\Big) \\ \text{(using inductive assumption)} \quad &\le e^{C|s_k^{p+1}|}\Big(\big|u_k^{p+1}(a_k^p) - u_k^p(a_k^p)\big| \\ &\qquad + Ce^{C\sum_{q=1}^p |s_k^q|} \sum_{q=1}^p \big|u_k^q(a_k^{q-1}) - u_k^{q-1}(a_k^{q-1})\big|\Big) \\ &\le Ce^{C\sum_{q=1}^{p+1} |s_k^q|} \sum_{q=1}^{p+1} \big|u_k^q(a_k^{q-1}) - u_k^{q-1}(a_k^{q-1})\big|,\end{aligned}$$

which is what we wanted to prove.
Assume now that (3.11) holds. Then we can use Lemma 3.7 and we get

$$\big\|v_k^p - \hat{v}_k^p\big\|_\infty, \ \big\|\sigma_k^p - \hat{\sigma}_k^p\big\|_\infty, \leq C|s_k^p| \sum_{q=1}^{p} \big|u_k^q(a_k^{q-1}) - u_k^{q-1}(a_k^{q-1})\big|,$$

which, together with (3.11) concludes the proof of the lemma. $\square$

LEMMA 3.9 (Change of the length of the curve). *Consider a curve $\gamma_k = (u_k, v_k, \sigma_k) = \Gamma_k(u^L, s)$. Fix any $\bar{\tau} \in \boldsymbol{I}(s)$ and a number $s' \in \mathbb{R}$ such that $|s'| \leq |s - \bar{\tau}|$. Consider the curve $\gamma_k' = (u_k', v_k' \sigma_k') := \Gamma_k(u_k(\bar{\tau}), s')$ and assume that γ_k' is defined on $\bar{\tau} + \boldsymbol{I}(s')$. Then it holds*

$$\big\|u_k - u_k'\big\|_{L^\infty(\tau + \boldsymbol{I}(s'))}, \ \big\|v_k - v_k'\big\|_{L^\infty(\tau + \boldsymbol{I}(s'))}, \ \big\|\sigma_k - \sigma_k'\big\|_{L^1(\bar{\tau} + \boldsymbol{I}(s'))} \leq 2\Big(\big|v_k(\bar{\tau})\big| + \big|v_k(\bar{\tau} + s')\big|\Big).$$

PROOF. We assume that $0 < \bar{\tau} < \bar{\tau} + s' < s$. All the other cases can be treated similarly. We have

$$\begin{cases} u_k(\tau) = u_k(\bar{\tau}) + \int_{\bar{\tau}}^{\tau} \tilde{r}_k\big(\gamma(\varsigma)\big) d\varsigma \\ v_k(\tau) = f_k(\tau) - \operatorname{conv}_{[0,s]} f_k(\tau) \\ \sigma_k(\tau) = D \operatorname{conv}_{[0,s]} f_k(\tau). \end{cases}$$

where $f_k(\tau) = \int_0^\tau \tilde{\lambda}_k(\gamma(\varsigma)) d\varsigma$. is the reduced flux associated to γ_k. We have to compute $D\big(\gamma_k|_{[\bar{\tau} + \mathbf{I}(s')]}, \gamma_k'\big)$. By Contraction Mapping Theorem,

$$D\big(\gamma_k|_{[\bar{\tau} + \mathbf{I}(s')]}, \gamma_k'\big) \leq 2D\Big(\gamma_k|_{[\bar{\tau} + \mathbf{I}(s')]}, \mathcal{T}\big(\gamma_k|_{[\bar{\tau} + \mathbf{I}(s')]}\big)\Big).$$

Set $\hat{\gamma}_k := (\hat{u}_k, \hat{v}_k, \hat{\sigma}_k) := \mathcal{T}\big(\gamma_k|_{[\bar{\tau} + \mathbf{I}(s')]}\big)$. Then

$$\begin{cases} \hat{u}_k(\tau) = u_k(\bar{\tau}) + \int_{\bar{\tau}}^{\tau} \tilde{r}_k\big(\gamma_k(\varsigma)\big) d\varsigma \\ \hat{v}_k(\tau) = \hat{f}_k(\tau) - \operatorname{conv}_{[\bar{\tau}, \bar{\tau} + s']} \hat{f}_k(\tau) \\ \hat{\sigma}_k(\tau) = D \operatorname{conv}_{[\bar{\tau}, \bar{\tau} + s']} \hat{f}_k(\tau), \end{cases}$$

where $\hat{f}_k(\tau) := f_k(\bar{\tau}) + \int_{\bar{\tau}}^{\tau} \tilde{\lambda}_k\big(\gamma_k(\varsigma)\big) d\varsigma = f_k(\tau)$ is the reduced flux computed on the curve $\gamma_k|_{\bar{\tau} + \mathbf{I}(s')}$ (recall that we can always add a constant to the reduced flux). Hence $\hat{u}_k(\tau) = u_k(\tau)$ for any $\tau \in [\bar{\tau}, \bar{\tau} + s']$. For the v-component we have

$$\begin{aligned} \big|\hat{v}_k(\tau) - v_k(\tau)\big| &\leq \Big|\Big(f_k(\tau) - \operatorname*{conv}_{[\bar{\tau}, \bar{\tau} + s']} f_k(\tau)\Big) - \Big(f_k(\tau) - \operatorname*{conv}_{[0,s]} f_k(\tau)\Big)\Big| \\ &= \Big|\operatorname*{conv}_{[\bar{\tau}, \bar{\tau} + s']} f_k(\tau) - \operatorname*{conv}_{[0,s]} f_k(\tau)\Big| \\ &= \Big|\operatorname*{conv}_{[\bar{\tau}, \bar{\tau} + s']} f_k(\tau) - \operatorname*{conv}_{[\bar{\tau},s]} f_k(\tau)\Big| + \Big|\operatorname*{conv}_{[\bar{\tau},s]} f_k(\tau) - \operatorname*{conv}_{[0,s]} f_k(\tau)\Big| \\ \text{(by Proposition 1.6)} \quad &\leq \big|f_k(\bar{\tau}) - \operatorname*{conv}_{[0,s]} f_k(\bar{\tau})\big| + \big|f_k(s') - \operatorname*{conv}_{[0,s]} f_k(s')\big| \\ &= \big|v_k(\bar{\tau})\big| + \big|v_k(s')\big|. \end{aligned}$$

Similarly, using again Proposition 1.6,

$$\big|\hat{\sigma}_k(\tau) - \sigma_k(\tau)\big| \leq \big|f_k(\bar{\tau}) - \operatorname*{conv}_{[0,s]} f_k(\bar{\tau})\big| + \big|f_k(s') - \operatorname*{conv}_{[0,s]} f_k(s')\big| = \big|v_k(\bar{\tau})\big| + \big|v_k(s')\big|.$$

Hence

$$D\big(\gamma_k|_{[\bar{\tau} + \mathbf{I}(s')]}, \gamma_k'\big) \leq 2D\Big(\gamma_k|_{[\bar{\tau} + \mathbf{I}(s')]}, \mathcal{T}\big(\gamma_k|_{[\bar{\tau} + \mathbf{I}(s')]}\big)\Big) \leq 2\Big(\big|v_k(\bar{\tau})\big| + \big|v_k(\bar{\tau} + s')\big|\Big). \quad \square$$

COROLLARY 3.10. *In the same hypothesis as in the previous lemma,*

$$\|u_k - u'_k\|_{L^\infty(\tau+\boldsymbol{I}(s'))}, \ \|v_k - v'_k\|_{L^\infty(\tau+\boldsymbol{I}(s'))}, \ \|\sigma_k - \sigma'_k\|_{L^1(\bar\tau+\boldsymbol{I}(s'))} \le \mathcal{O}(1)\Big(|\bar\tau| + |s-(\bar\tau+s')|\Big).$$

PROOF. The proof is an immediate consequence of the fact that v_k is Lipschitz with Lipschitz constant depending only on the flux F and $v_k(0) = v_k(s) = 0$. $\square$

The following lemma is a generalization of the previous one.

LEMMA 3.11 (Creation of waves). *Let $s, s' \in \mathbb{R}$. Assume that they have the same sign and $|s'| \le |s|$. Let $\Theta : \boldsymbol{I}(s') \to \boldsymbol{I}(s)$ be a piecewise affine map with slope equal to 1. Let k be a fixed family. Let $\gamma_k = (u_k, v_k, \sigma_k) := \Gamma_k(u^L, s)$ and $\gamma'_k = (u'_k, v'_k, \sigma'_k) := \Gamma_k(u^L, s')$. Then*

$$D\big(\gamma'_k, \gamma_k \circ \Theta\big) \le \mathcal{O}(1)|s-s'|.$$

PROOF. By the Contraction Mapping Principle,

$$D(\gamma'_k, \gamma_k\circ\Theta) \le 2D\big(\mathcal{T}(\gamma_k\circ\Theta), \gamma_k\circ\Theta\big).$$

We have

$$\begin{cases} u_k(\tau) = u^L + \int_{\bar\tau}^\tau \tilde r_k\big(\gamma_k(\varsigma)\big)d\varsigma \\ v_k(\tau) = f_k(\tau) - \operatorname{conv}_{[0,s]} f_k(\tau) \\ \sigma_k(\tau) = D\operatorname{conv}_{[0,s]} f_k(\tau), \end{cases} \qquad \begin{cases} u'_k(\tau) = u^L + \int_0^\tau \tilde r_k\big(\gamma'_k(\varsigma)\big)d\varsigma \\ v'_k(\tau) = f'_k(\tau) - \operatorname{conv}_{[0,s']} f'_k(\tau) \\ \sigma'_k(\tau) = D\operatorname{conv}_{[0,s']} f'_k(\tau), \end{cases}$$

where $f_k(\tau) = \int_0^\tau \tilde\lambda_k(\gamma_k(\varsigma))d\varsigma$ is the reduced flux associated to γ_k and $f'_k(\tau) = \int_0^\tau \tilde\lambda_k(\gamma'_k(\varsigma))d\varsigma$ is the reduced flux associated to γ'_k. Set $\tilde\gamma_k = (\tilde u_k, \tilde v_k, \tilde\sigma_k) := \mathcal{T}(\gamma_k\circ\Theta)$ and denote by

$$\tilde f_k(\tau) := \int_0^\tau \tilde\lambda_k\big(\gamma_k\circ\Theta(\varsigma)\big)d\varsigma$$

the associated reduced flux. We have

$$\begin{cases} \tilde u_k(\tau) = u^L + \int_0^\tau \tilde r_k\big(\gamma_k\circ\Theta(\varsigma)\big)d\varsigma \\ \tilde v_k(\tau) = \tilde f_k(\tau) - \operatorname{conv}_{[0,s']} \tilde f_k(\tau) \\ \tilde\sigma_k(\tau) = D\operatorname{conv}_{[0,s']} \tilde f_k(\tau). \end{cases}$$

Denote by Θ^{-1} the pseudo-inverse of Θ which turns out to be a Lipschitz map. Observe that

$$\text{if } \Theta\circ\Theta^{-1}(\tau) \neq \tau, \text{ then } D\Theta^{-1}(\tau) = 0. \tag{3.12}$$

We can thus perform the following computation:

$$\begin{aligned} \tilde u_k(\tau) - u_k\big(\Theta(\tau)\big) &= \int_0^\tau \tilde r_k\big(\gamma_k(\Theta(\upsilon))\big)d\upsilon - \int_0^{\Theta(\tau)} \tilde r_k\big(\gamma_k(\varsigma)\big)d\varsigma \\ &\text{(making the change of variable } \upsilon = \Theta^{-1}(\varsigma) \text{ and using (3.12))} \\ &= \int_0^{\Theta(\tau)} \tilde r_k\big(\gamma_k(\varsigma)\big)(\Theta^{-1})(\varsigma)d\varsigma - \int_0^{\Theta(\tau)} \tilde r_k\big(\gamma_k(\varsigma)\big)d\varsigma \\ &\le \int_0^{\Theta(\tau)} \Big|\tilde r_k\big(\gamma'_k(\varsigma)\big)\Big|\Big|1-(\Theta^{-1})(\varsigma)\Big|d\varsigma \\ &\le \mathcal{O}(1)|s-s'|. \end{aligned}$$

Similarly

$$
\begin{aligned}
\|D\tilde{f} - f\|_{L^1(\mathbf{I}(s'))} &= \int_0^{s'} |D\tilde{f}_k(\tau) - f_k(\tau)| d\tau \\
&= \int_0^{s'} \Big|\tilde{\lambda}_k\big(\gamma_k(\Theta(\tau))\big) - \tilde{\lambda}_k\big(\gamma_k(\tau)\big)\Big| d\tau \\
&\leq \mathcal{O}(1) \int_0^{s'} |\Theta(\tau) - \tau| d\tau \\
&\leq \mathcal{O}(1)|s'||s - s'|
\end{aligned}
\tag{3.13}
$$

and thus

$$
\|\tilde{f} - f\|_{L^\infty(I(s'))} \leq \mathcal{O}(1)|s'||s - s'|. \tag{3.14}
$$

Therefore, for $\tau \in \mathbf{I}(s')$, we have

$$
\begin{aligned}
|\tilde{v}_k(\tau) - v(\Theta(\tau))| &= \left| \Big(\tilde{f}_k(\tau) - \operatorname*{conv}_{[0,s']} \tilde{f}_k(\tau)\Big) - \Big(f_k(\Theta(\tau)) - (\operatorname*{conv}_{[0,s]} f)(\Theta(\tau))\Big) \right| \\
&\leq \Big|\tilde{f}_k(\tau) - f_k(\Theta(\tau))\Big| + \Big| \operatorname*{conv}_{[0,s']} \tilde{f}_k(\tau) - (\operatorname*{conv}_{[0,s]} f)(\Theta(\tau)) \Big| \\
&\leq \big|\tilde{f}_k(\tau) - f_k(\tau)\big| + \big|f_k(\tau) - f_k(\Theta(\tau))\big| + \big| \operatorname*{conv}_{[0,s']} \tilde{f}_k(\tau) - \operatorname*{conv}_{[0,s']} f_k(\tau) \big| \\
&\quad + \big| \operatorname*{conv}_{[0,s']} f_k(\tau) - \operatorname*{conv}_{[0,s]} f_k(\tau) \big| + \big| \operatorname*{conv}_{[0,s]} f_k(\tau) - \operatorname*{conv}_{[0,s]} f_k((\Theta(\tau)) \big| \\
&\leq \mathcal{O}(1)|s - s'|,
\end{aligned}
$$

where in the last inequality we have used (3.14), Corollary 1.10, Proposition 1.11 and the fact that $|\Theta(\tau) - \tau| \leq |s - s'|$. A similar computation shows that

$$
\big\|\tilde{\sigma}_k - \sigma_k \circ \Theta\big\|_1 \leq \mathcal{O}(1)|s - s'|,
$$

thus concluding the proof of the Lemma. □

Lemma 3.12 (Transversal interaction). *Let $u^M = T^k_{s_k}(u^L)$, $u^R = T^h_{s_h}(u^M)$. Set $\hat{u}^M := T^h_{s_h}(u^L)$, $\hat{u}^R := T^k_{s_k}(\hat{u}^M)$ and*

$$
\begin{aligned}
\gamma_k &= (u_k, v_k, \sigma_k) := \Gamma_k(u^L, s_k), & \gamma_h &= (u_h, v_h, \sigma_h) := \Gamma_h(u^M, s_h), \\
\widehat{\gamma}_h &= (\hat{u}_h, \hat{v}_h, \hat{\sigma}_h) := \Gamma_h(u^L, s_h), & \widehat{\gamma}_k &= (\hat{u}_k, \hat{v}_k, \hat{\sigma}_k) := \Gamma_k(\hat{u}^M, s_k).
\end{aligned}
$$

Denote by $f_k, \hat{f}_k, f_h, \hat{f}_h$ the reduced fluxes associated to the curves $\gamma_k, \hat{\gamma}_k, \gamma_h, \hat{\gamma}_h$ respectively. Then it holds

$$
\|u_h - \hat{u}_h\|_\infty \leq \mathcal{O}(1)|s_k|, \qquad \|u_k - \hat{u}_k\|_\infty \leq \mathcal{O}(1)|s_h|, \tag{3.15a}
$$

$$
\|v_h - \hat{v}_h\|_\infty \leq \mathcal{O}(1)|s_k||s_h|, \qquad \|v_k - \hat{v}_k\|_\infty \leq \mathcal{O}(1)|s_k||s_h|, \tag{3.15b}
$$

$$
\|\sigma_h - \hat{\sigma}_h\|_1 \leq \mathcal{O}(1)|s_k||s_h|, \qquad \|\sigma_k - \hat{\sigma}_k\|_1 \leq \mathcal{O}(1)|s_k||s_h|, \tag{3.15c}
$$

$$
\left\| \frac{d^2 f_k}{d\tau^2} - \frac{d^2 \hat{f}_k}{d\tau^2} \right\|_1 \leq \mathcal{O}(1)|s_k||s_h|, \qquad \left\| \frac{d^2 f_h}{d\tau^2} - \frac{d^2 \hat{f}_h}{d\tau^2} \right\|_1 \leq \mathcal{O}(1)|s_k||s_h|,
$$

and

$$
|u^R - \hat{u}^R| \leq \mathcal{O}(1)|s_k||s_h|. \tag{3.16}
$$

PROOF. Inequalities (3.15) are direct consequence of Lemma 3.7 and the fact that

$$\big|u^M - u^L\big| \le \mathcal{O}(1)|s_k|, \qquad\qquad \big|\hat{u}^M - u^L\big| \le \mathcal{O}(1)|s_h|.$$

Let us now prove inequality (3.16). We have

$$\begin{aligned} u^R &= u^L + \int_0^{s_k} \tilde{r}_k(\gamma_k(\varsigma))d\varsigma + \int_0^{s_h} \tilde{r}_h(\gamma_h(\varsigma))d\varsigma, \\ \hat{u}^R &= u^L + \int_0^{s_h} \tilde{r}_h(\widehat{\gamma}_h(\varsigma))d\varsigma + \int_0^{s_k} \tilde{r}_k(\widehat{\gamma}_k(\varsigma))d\varsigma. \end{aligned}$$

Hence

$$\begin{aligned} |u^R - \hat{u}^R| &\le \int_0^{s_k} \big|\tilde{r}_k(\gamma_k(\varsigma)) - \tilde{r}_k(\widehat{\gamma}_k(\varsigma))\big|d\varsigma + \int_0^{s_h} \big|\tilde{r}_h(\gamma_h(\varsigma)) - \tilde{r}_h(\widehat{\gamma}_h(\varsigma))\big|d\varsigma \\ &\le \mathcal{O}(1)\bigg[\int_0^{s_k} \Big(|u_k(\varsigma) - \hat{u}_k(\varsigma)| + |v_k(\varsigma) - \hat{v}_k(\varsigma)| + \big|\sigma_k(\varsigma) - \hat{\sigma}_k(\varsigma)\big|\Big)d\varsigma \\ &\qquad\qquad + \int_0^{s_h} \Big(|u_h(\varsigma) - \hat{u}_h(\varsigma)| + |v_h(\varsigma) - \hat{v}_h(\varsigma)| + \big|\sigma_h(\varsigma) - \hat{\sigma}_h(\varsigma)\big|\Big)d\varsigma\bigg] \\ &\le \mathcal{O}(1)\Big[|s_k|\|u_k - \hat{u}_k\|_\infty + |s_k|\|v_k - \hat{v}_k\|_\infty + \|\sigma_k - \hat{\sigma}_k\|_1 \\ &\qquad\qquad |s_h|\|u_h - \hat{u}_h\|_\infty + |s_h|\|v_h - \hat{v}_h\|_\infty + \|\sigma_h - \hat{\sigma}_h\|_1\Big] \\ &\le \mathcal{O}(1)|s_k||s_h|, \end{aligned}$$

where, in the last inequality, we have used (3.15a)-(3.15c). □

In the previous lemma we considered a transversal interaction between only two curves. The following situation describes a more general transversal interaction, when many curves of many families are present. Let γ_k^p, $p = 1, \dots, P$, $k = 1, \dots, N$ be a collection of NP exact curves, with $P \in \mathbb{N} \setminus \{0\}$. Denote by $\gamma_k^p = (u_k^p, v_k^p, \sigma_k^p)$ the components of γ_k^p and by f_k^p the associated reduced fluxes. Assume that

(1) for any p, γ_k^p is an exact curve of the k-th family with length s_k^p;
(2) the starting point of the first curves γ_1^1 is u^L;
(3) the curves $\{\gamma_k^p\}_k^p$ are consecutive w.r.t. the order

$$(p,k) \text{ precedes } (p',k') \iff p < p' \text{ or } p = p' \text{ and } k < k'.$$

Consider now another collection of NP curves $\{\tilde{\gamma}_k^p\}_k^p$, $p = 1, \dots, P$, $k = 1, \dots, N$. Denote by $\tilde{\gamma}_k^p = (\tilde{u}_k^p, \tilde{v}_k^p, \tilde{\sigma}_k^p)$ the components of $\tilde{\gamma}_k^p$ and by $\tilde{f}_k^p$ the associated reduced fluxes. Assume that

(1) for any p, $\tilde{\gamma}_k^p$ is an exact curve of the k-th family with length s_k^p;
(2) the starting point of the first curves $\tilde{\gamma}_1^1$ is u^L;
(3) the curves $\{\tilde{\gamma}_k^p\}_k^p$ are consecutive w.r.t. the order

$$(p,k) \text{ precedes } (p',k') \iff k < k' \text{ or } k = k' \text{ and } p < p'.$$

Observe that the curves $\{\tilde{\gamma}_k^p\}_k^p$ are obtained from the curves $\{\gamma_k^p\}_k^p$ after all the transversal interactions took place. For any k and p assume also that γ_k^p, f_k^p, $\tilde{\gamma}_k^p$, $\tilde{f}_k^p$ are defined on the same interval $\mathbf{I}(s_k^p)$. Then the following corollary holds.

COROLLARY 3.13 (Many transversal interactions). *It holds*

$$\left.\begin{array}{r}\displaystyle\sum_{k=1}^{N}\sum_{p=1}^{P}\big\|u_k^p-\tilde u_k^p\big\|_1\\[2ex] \displaystyle\sup_{k=1,\dots N}\sup_{p=1,\dots,P}\big\|v_k^p-\tilde v_k^p\big\|_\infty\\[2ex] \displaystyle\sum_{k=1}^{N}\sum_{p=1}^{P}\big\|\sigma_k^p-\tilde \sigma_k^p\big\|_1\\[2ex] \displaystyle\sum_{k=1}^{N}\sum_{p=1}^{P}\|D^2f_k^p-D^2\tilde f_k^p\|_1\end{array}\right\}\leq \mathcal{O}(1)\sum_{p<p'}\sum_{k>k'}|s_k^p||s_{k'}^{p'}|$$

and

$$\big|u_k^p(s_k^p)-\tilde u_k^p(s_k^p)\big|\leq\leq\mathcal{O}(1)\sum_{q<q'}\sum_{h>h'}|s_h^q||s_{h'}^{q'}|, \tag{3.17}$$

where $q,q'\in\{1,\dots,P\}$ *and* $h,h'\in\{1,\dots,N\}$.

Notice that we estimate the L^1 distance of the u-components and not their L^∞ distance.

PROOF. The proof can be obtained applying several times Lemma 3.12 and using Lemma 3.7. □

We conclude this section with the following lemma, which analyzes the interaction between many exact curves of the same family and same sign.

LEMMA 3.14 (Same family interactions). *Let k be a fixed family. Let $\{\gamma_k^p\}_p$, $p=1,\dots,P$, be a family of P consecutive curves of the k-th family. Denote by $\gamma_k^p=(u_k^p,v_k^p,\sigma_k^p)$ the components of γ_k^p and let f_k^p be the associated reduced flux. Assume that the length of γ_k^p is s^p and set $s:=\sum_p s^p$. Assume that all the s^p have the same sing. Suppose also that ($\star$) is satisfied. Let $\gamma_k=(u_k,v_k,\sigma_k):=\Gamma_k\Big(u^1(0),s\Big)$ and let f_k be its associated reduced flux. Then, if all the s^p are positive,*

$$\left.\begin{array}{r}\displaystyle D\bigg(\gamma_k,\bigcup_{p=1}^{P}\gamma_k^p\bigg),\\[2ex] \displaystyle\|D^2f_k-D^2\bigcup_{p=1}^{P}f_k^p\|_{L^1(\boldsymbol{I}(s))}\end{array}\right\}\leq\mathcal{O}(1)\bigg\|D\operatorname*{conv}_{\boldsymbol{I}(s)}\bigcup_{p=1}^{P}f_k^p-D\bigcup_{p=1}^{P}\operatorname*{conv}_{\boldsymbol{I}(s^p)}f_k^p\bigg\|_{L^1(\boldsymbol{I}(s))}$$

and

$$\left.\begin{array}{r}\displaystyle D\bigg(\gamma,\bigcup_{p=1}^{P}\gamma_k^p\bigg),\\[2ex] \displaystyle\|D^2f_k-D^2\bigcup_{p=1}^{P}f_k^p\|_{L^1(\boldsymbol{I}(s))}\end{array}\right\}\leq\mathcal{O}(1)\bigg\|D\operatorname*{conv}_{\boldsymbol{I}(s)}f_k-D\bigcup_{p=1}^{P}\operatorname*{conv}_{\boldsymbol{I}(s^p)}f_k\bigg\|_{L^1(\boldsymbol{I}(s))}.$$

If all the s^p are negative, a completely similar result holds, with the concave envelope, instead of the convex one.

PROOF. We assume that $s^p > 0$ for any $p = 1, \dots, P$, the negative case being completely similar. Let us first prove that

$$D\Big(\gamma_k, \bigcup_{p=1}^P \gamma_k^p\Big) \le \mathcal{O}(1) \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^P f_k^p - D \bigcup_{p=1}^P \operatorname*{conv}_{\mathbf{I}(s^p)} f_k^p \Big\|_{L^1(\mathbf{I}(s))} \tag{3.18}$$

We have

$$\begin{cases} u_k^p(\tau) = u^L + \displaystyle\int_0^\tau \tilde r_k(\gamma_k^p(\varsigma)) d\varsigma, \\ v_k^p(\tau) = f_k^p(\tau) - \operatorname*{conv}_{[0,s^p]} f_k^p(\tau), \\ \sigma_k^p(\tau) = D \operatorname*{conv}_{[0,s^p]} f_k^p(\tau), \end{cases}$$

Set $\hat\gamma_k = (\hat u_k, \hat v_k, \hat\sigma_k) := \mathcal{T}\Big(\bigcup_{p=1}^P \gamma_k^p\Big)$:

$$\begin{cases} \hat u_k(\tau) = u^L + \displaystyle\int_0^\tau \tilde r_k\Big(\bigcup_{p=1}^P \gamma_k^p(\varsigma)\Big) d\varsigma \\ \hat v_k(\tau) = \displaystyle\bigcup_{p=1}^P f_k^p(\tau) - \operatorname*{conv}_{[0,s]} \bigcup_{p=1}^P f_k^p(\tau), \\ \hat\sigma_k(\tau) = D \operatorname*{conv}_{[0,s]} \displaystyle\bigcup_{p=1}^P f_k^p. \end{cases}$$

Therefore

$$\Big\| \bigcup_{p=1}^P u_k^p - \hat u_k \Big\|_\infty = 0, \qquad \Big\| \bigcup_{p=1}^P v_k^p - \hat v_k \Big\|_\infty \le \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} f_k - D \bigcup_{p=1}^P \operatorname*{conv}_{\mathbf{I}(s^p)} f_k \Big\|_{L^1(\mathbf{I}(s))}.$$

and

$$\Big\| \bigcup_{p=1}^P \sigma_k^p - \hat\sigma_k \Big\|_1 = \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} f_k - D \bigcup_{p=1}^P \operatorname*{conv}_{\mathbf{I}(s^p)} f_k \Big\|_{L^1(\mathbf{I}(s))}.$$

Hence, by the Contraction Mapping Theorem,

$$D\Big(\bigcup_{p=1}^P \gamma_k^p, \gamma_k\Big) \le 2D\Big(\bigcup_{p=1}^P \gamma_k^p, \hat\gamma\Big) \le 4 \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} f_k - D \bigcup_{p=1}^P \operatorname*{conv}_{\mathbf{I}(s^p)} f_k \Big\|_{L^1(\mathbf{I}(s))}.$$

The inequality

$$\|D^2 f_k - D^2 \bigcup_{p=1}^P f_k^p\|_{L^1(\mathbf{I}(s))} \le \mathcal{O}(1) \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^P f_k^p - D \bigcup_{p=1}^P \operatorname*{conv}_{\mathbf{I}(s^p)} f_k^p \Big\|_{L^1(\mathbf{I}(s))} \tag{3.19}$$

is a consequence of Lemma 3.6, Proposition 1.11 and (3.18).

To prove the second part of the statement, notice that

$$\begin{aligned}
&\Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^{P} f_k^p - D \bigcup_{p=1}^{P} \operatorname*{conv}_{\mathbf{I}(s^p)} f_k^p \Big\|_{L^1(\mathbf{I}(s))} \\
&\qquad \leq \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^{P} f_k^p - D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^{P} f_k \Big\|_1 + \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^{P} f_k - D \bigcup_{p=1}^{P} \operatorname*{conv}_{\mathbf{I}(s^p)} f_k \Big\|_1 \\
&\qquad\qquad + \Big\| D \bigcup_{p=1}^{P} \operatorname*{conv}_{\mathbf{I}(s^p)} f_k - D \bigcup_{p=1}^{P} \operatorname*{conv}_{\mathbf{I}(s^p)} f_k^p \Big\|_1 \\
\text{(by Proposition 1.11)}\quad &\qquad \leq 2 \Big\| D \bigcup_{p=1}^{P} f_k^p - D f_k \Big\|_1 + \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^{P} f_k - D \bigcup_{p=1}^{P} \operatorname*{conv}_{\mathbf{I}(s^p)} f_k \Big\|_1 \\
&\qquad \leq 2|s| \Big\| D^2 \bigcup_{p=1}^{P} f_k^p - D^2 f_k \Big\|_1 + \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^{P} f_k - D \bigcup_{p=1}^{P} \operatorname*{conv}_{\mathbf{I}(s^p)} f_k \Big\|_1
\end{aligned}$$

Therefore, using (3.19), we get

$$\Big\| D^2 \bigcup_{p=1}^{P} f_k^p - D^2 f_k \Big\|_1 \leq 2|s| \Big\| D^2 \bigcup_{p=1}^{P} f_k^p - D^2 f_k \Big\|_1 + \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^{P} f_k - D \bigcup_{p=1}^{P} \operatorname*{conv}_{\mathbf{I}(s^p)} f_k \Big\|_1$$

and thus, if $|s| < 1/2$,

$$\Big\| D^2 \bigcup_{p=1}^{P} f_k^p - D^2 f_k \Big\|_1 \leq \frac{1}{(1-2|s|)} \Big\| D \operatorname*{conv}_{\mathbf{I}(s)} \bigcup_{p=1}^{P} f_k - D \bigcup_{p=1}^{P} \operatorname*{conv}_{\mathbf{I}(s^p)} f \Big\|_1,$$

which together with (3.18) and (3.19) concludes the proof of the lemma. □

We conclude this section with the following lemma which shows that the exact curves are stable under sequential convergence.

LEMMA 3.15. *Let $u_L^n \in \mathbb{R}^N$, $n \in \mathbb{N}$ be a sequence of points such that $u_L^n \to u_L$ as $n \to \infty$. Let s^n, $n \in \mathbb{N}$, be a sequence of numbers such that $s^n \to s$ as $n \to \infty$. Then*

$$\Gamma_k(u_L^n, s_N) = (u_k^n, v_k^n, \sigma_k^n) \to \Gamma_k(u_L, s) = (u_k, v_k, \sigma_k) \ \text{as } n \to \infty$$

in the sense that

$$\|u_k^n - u_k\|_{L^\infty(\boldsymbol{I}(s^n)\cap \boldsymbol{I}(s))} \to 0, \quad \|v_k^n - v_k\|_{L^\infty(\boldsymbol{I}(s^n)\cap \boldsymbol{I}(s))} \to 0, \quad \|\sigma_k^n - \sigma_k\|_{L^1(\boldsymbol{I}(s^n)\cap \boldsymbol{I}(s))} \to 0$$

as $n \to \infty$.

PROOF. The proof follows easily from Lemma 3.7 and Corollary 3.10. □

3.2. Estimates for two merging Riemann problems

This section is devoted to prove the *local* part of the proof of Theorem A, as explained in the introduction of this chapter. In particular we will consider two contiguous Riemann problems (u^L, u^M), (u^M, u^R) which are merging, producing the Riemann problem (u^L, u^R) and we will introduce a *global amount of interaction* $\mathtt{A}$, which bounds

(1) the L^1-distance between the speed of the wavefronts before and after the interaction, i.e. the σ-component of the elementary curves;
(2) the L^1-distance between the second derivatives of the reduced fluxes, before and after the interaction.

This is done in Theorem 3.18.

3.2.1. Description of the reference situation. We first describe the reference situation we are going to study now and we introduce some notations. Consider two contiguous Riemann problem

$$u^M = T^N_{s'_N} \circ \cdots \circ T^1_{s'_1} u^L, \qquad u^R = T^N_{s''_N} \circ \cdots \circ T^1_{s''_1} u^M, \tag{3.20}$$

and the Riemann problem obtained joining them,

$$u^R = T^N_{s_N} \circ \cdots \circ T^1_{s_1} u^L.$$

In particular the incoming curves are

$$\begin{aligned} \gamma'_1 = (u'_1, v'_1, \sigma'_1) &:= \Gamma_1(u^L, s'_1), \quad \gamma'_k = (u'_k, v'_k, \sigma'_k) := \Gamma_k\big(u'_{k-1}(s'_{k-1}), s'_k\big) \quad \text{for } k = 2, \dots, n, \\ \gamma''_1 = (u''_1, v''_1, \sigma''_1) &:= \Gamma_1(u^M, s''_1), \quad \gamma''_k = (u''_k, v''_k, \sigma''_k) := \Gamma_k\big(u''_{k-1}(s''_{k-1}), s''_k\big) \quad \text{for } k = 2, \dots, n, \end{aligned} \tag{3.21}$$

while the outcoming ones are

$$\begin{aligned} \gamma_1 = (u_1, v_1, \sigma_1) &:= \Gamma_1(u^L, s_1), \\ \gamma_k = (u_k, v_k, \sigma_k) &:= \Gamma_k\big(u_{k-1}(s_{k-1}), s_k\big) \quad \text{for } k = 2, \dots, n. \end{aligned} \tag{3.22}$$

We will denote by f'_k, f''_k, f_k the reduced fluxes associated to $\gamma'_k, \gamma''_k, \gamma_k$ respectively; we assume also that for each $k = 1, \dots, N$, the two curves γ'_k, γ''_k satisfies the assumption ($\star$).
Fix now an index $k \in \{1, \dots n\}$ and consider the points (Figure 1)

$$\begin{aligned} u^L_1 &:= u^L, \qquad u^L_k := T^{k-1}_{s''_{k-1}} \circ T^{k-1}_{s'_{k-1}} \circ \cdots \circ T^1_{s''_1} \circ T^1_{s'_1} u^L, \quad k \geq 2 \\ u^M_k &:= T^k_{s'_k} u^L_k, \quad u^R_k := T^k_{s''_k} u^M_k, \qquad\qquad k = 1, \dots, n. \end{aligned}$$

By definition, the Riemann problem between u^L_k and u^M_k is solved by a wavefront of the k-th family with strength s'_k and the Riemann problem between u^M_k and u^R_k is solved by a wavefront of the k-th family with strength s''_k. Denote by $\tilde{\gamma}'_k = (\tilde{u}'_k, \tilde{v}'_k, \tilde{\sigma}'_k)$ the curve which solves the Riemann problem $[u^L_k, u^M_k]$ and by $\tilde{f}'_k$ the associated reduced flux.
Similarly, let $\tilde{\gamma}''_k = (\tilde{u}''_k, \tilde{v}''_k, \tilde{\sigma}''_k)$ be the curve solving the Riemann problem $[u^M_k, u^R_k]$ and let $\tilde{f}''_k$ be the associated reduced flux. We assume that, for each k, the pair of curves $\tilde{\gamma}'_k, \tilde{\gamma}''_k$ satisfies the assumption ($\star$).

3.2.2. Statement of the theorem. We have already introduced in Section 2.2 the notions of transversal amount of interaction, amount of creation, amount of cancellation and cubic amount of interaction. We now define a new quantity, namely the *quadratic amount of interaction*, which will be used to bound the L^1-norm of the difference of speed between incoming and outgoing wavefronts.

DEFINITION 3.16. If $s'_k s''_k \geq 0$, we define the *quadratic amount of interaction of the k-family* associated to the two Riemann problems (3.20) by

$$\mathtt{A}^{\mathrm{quadr}}_k(u^L, u^M, u^R)$$
$$:= \begin{cases} \Big\| D\,\mathrm{conv}_{\mathbf{I}(s'+s'')} f_k - \big(D\,\mathrm{conv}_{\mathbf{I}(s')} f'_k \cup D\,\mathrm{conv}_{\mathbf{I}(s'')} f''_k\big)\Big\|_1 & \text{if } s'_k > 0,\ s''_k > 0, \\ \Big\| D\,\mathrm{conc}_{\mathbf{I}(s'+s'')} f_k - \big(D\,\mathrm{conc}_{\mathbf{I}(s')} f'_k \cup D\,\mathrm{conc}_{\mathbf{I}(s'')} f''_k\big)\Big\|_1 & \text{if } s'_k < 0,\ s''_k < 0, \\ 0 & \text{if } s'_k s''_k \leq 0. \end{cases}$$

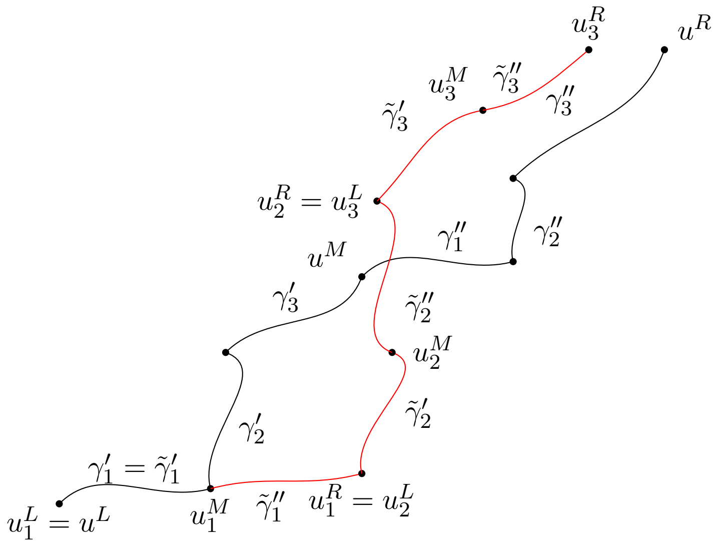

FIGURE 1. Elementary curves of two interacting Riemann problems before and after transversal interactions.

DEFINITION 3.17. We define the *global amount of interaction* associated to the two Riemann problems (3.20) as

$$\mathtt{A}(u^L, u^M, u^R) := \mathtt{A}^{\text{trans}}(u^L, u^M, u^R) + \sum_{h=1}^{N} \Big(\mathtt{A}_h^{\text{quadr}}(u^L, u^M, u^R) + \mathtt{A}_h^{\text{canc}}(u^L, u^M, u^R) + \mathtt{A}_h^{\text{cubic}}(u^L, u^M, u^R) \Big).$$

The main result of this section, which proves the local part of Theorem A, is the following.

THEOREM 3.18. *For any* $k = 1, \dots, N$,

- *if* $s_k' s_k'' \geq 0$, *then*

$$\left. \begin{aligned} & \big\| (\sigma_k' \cup \sigma_k'') - \sigma_k \big\|_{L^1(\boldsymbol{I}(s_k'+s_k'')\cap \boldsymbol{I}(s_k))} \\ & \bigg\| \bigg(\frac{d^2 f_k'}{d\tau^2} \cup \frac{d^2 f_k''}{d\tau^2} \bigg) - \frac{d^2 f_k}{d\tau^2} \bigg\|_{L^1(\boldsymbol{I}(s_k'+s_k'')\cap \boldsymbol{I}(s_k))} \end{aligned} \right\} \leq \mathcal{O}(1)\mathtt{A}(u^L, u^M, u^R);$$

- *if* $s_k' s_k'' < 0$, *then*

$$\left. \begin{aligned} & \big\| (\sigma_k' \vartriangle \sigma_k'') - \sigma_k \big\|_{L^1(\boldsymbol{I}(s_k'+s_k'')\cap \boldsymbol{I}(s_k))} \\ & \bigg\| \bigg(\frac{d^2 f_k'}{d\tau^2} \vartriangle \frac{d^2 f_k''}{d\tau^2} \bigg) - \frac{d^2 f_k}{d\tau^2} \bigg\|_{L^1(\boldsymbol{I}(s_k'+s_k'')\cap \boldsymbol{I}(s_k))} \end{aligned} \right\} \leq \mathcal{O}(1)\mathtt{A}(u^L, u^M, u^R).$$

See Section 1.1 for the definition of $\vartriangle$.

3.2.3. Proof of Theorem 3.18. To prove Theorem 3.18, we piece together all the estimates of the previous section, as follows. First of all we split the operation of "merging the two Riemann problems" into three steps:

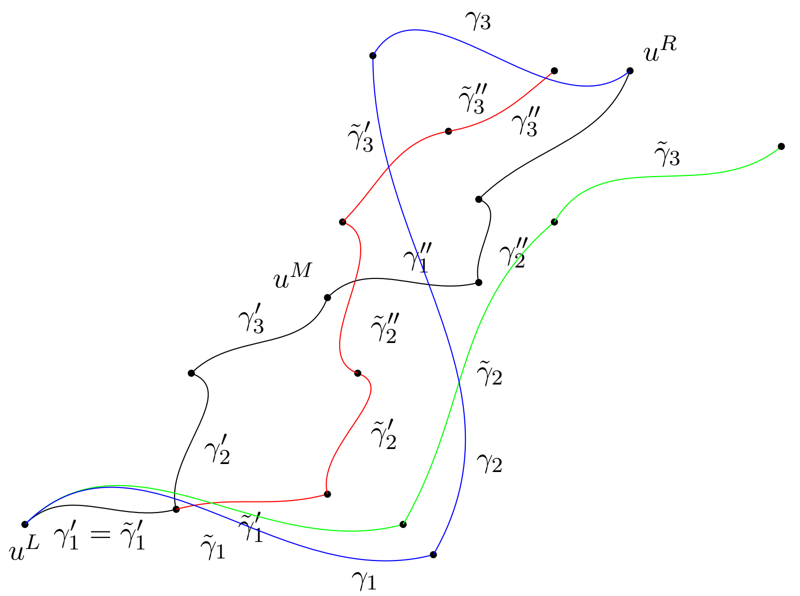

FIGURE 2. Elementary curves of two interacting Riemann problems before the interaction (black ones), after transversal interaction (red ones), after interaction/cancellation (collision) among wavefronts of the same family (green ones), after creation/cancellation (perturbation of the total variation) due to non-linearity (blue ones).

(1) first we pass from the collection of curves (3.21), i.e. the black ones in Figure 2, to the collection of curves

$$\begin{aligned} &\tilde\gamma_1' = (\tilde u_1', \tilde v_1', \tilde\sigma_1') := \gamma_1(u^L, s_1'), && \tilde\gamma_1'' = (\tilde u_1'', \tilde v_1'', \tilde\sigma_1'') := \gamma_1(u_1'(s_1'), s_1''), \\ &\tilde\gamma_k' = (\tilde u_k', \tilde v_k', \tilde\sigma_k') := \gamma_k\big(\tilde u_{k-1}''(s_{k-1}''), s_k'\big), && \tilde\gamma_k'' = (\tilde u_k'', \tilde v_k'', \tilde\sigma_k'') := \gamma_k\big(\tilde u_k'(s_k'), s_k''\big), \ k = 2, \dots, n; \end{aligned} \tag{3.23}$$

i.e. the red curves in Figure 2; this first step will be called *transversal interactions* and it will be studied in Lemma 3.19;

(2) as a second step, we let the curves of the same family interact, passing from the collection of red curves (3.23) to the collection of curves (green in Figure 2)

$$\begin{aligned} &\tilde\gamma_1 = (\tilde u_1, \tilde v_1, \tilde\sigma_1) := \gamma_1(u^L, s_1' + s_1''), \\ &\tilde\gamma_k = (\tilde u_k, \tilde v_k, \tilde\sigma_k) := \gamma_k\big(\tilde u_{k-1}(s_{k-1}' + s_{k-1}''), s_k' + s_k''\big), \ k = 2, \dots, n; \end{aligned} \tag{3.24}$$

this operation will be called *collision among waves of the same family* and it will be studied in Lemma 3.20;

(3) finally we pass from the collection of green curves (3.24) to the outcoming collection of curves (3.22), blue in Figure 2; this operation will be called *perturbation of the total variation due to nonlinearity*, and it will be studied in Lemma 3.21 and its Corollary 3.22.

Let us begin with the analysis of transversal interactions.

LEMMA 3.19. *For any $k = 1, \dots, N$, it holds*

$$\left.\begin{array}{l} \|\sigma_k' - \tilde{\sigma}_k'\|_{L^1(\boldsymbol{I}(s_k'))} \\ \|\sigma_k'' - \tilde{\sigma}_k''\|_{L^1(s_k' + \boldsymbol{I}(s_k''))} \\ \left\| \dfrac{d^2 f_k'}{d\tau^2} - \dfrac{d^2 \tilde{f}_k'}{d\tau^2} \right\|_{L^1(\boldsymbol{I}(s_k'))} \\ \left\| \dfrac{d^2 f_k''}{d\tau^2} - \dfrac{d^2 \tilde{f}_k''}{d\tau^2} \right\|_{L^1(s_k' + \boldsymbol{I}(s_k''))} \end{array}\right\} \leq \mathcal{O}(1) \mathtt{A}^{\mathrm{trans}}(u^L, u^M, u^R).$$

PROOF. The proof is an immediate application of Corollary 3.13 and Definition 2.5 of $\mathtt{A}^{\mathrm{trans}}(u^L, u^M, u^R)$. $\square$

Let us now analyze the collision among waves of the same family.

LEMMA 3.20. *For any $k = 1, \dots, N$,*

- *if $s_k' s_k'' \geq 0$, then*

$$\left.\begin{array}{l} \|(\tilde{\sigma}_k' \cup \tilde{\sigma}_k'') - \tilde{\sigma}_k\|_{L^1(\boldsymbol{I}(s_k' + s_k''))} \\ \left\| \left(\dfrac{d^2 \tilde{f}_k'}{d\tau^2} \cup \dfrac{d^2 \tilde{f}_k''}{d\tau^2} \right) - \dfrac{d^2 \tilde{f}_k}{d\tau^2} \right\|_{L^1(\boldsymbol{I}(s_k' + s_k''))} \end{array}\right\} \leq \mathcal{O}(1) \left[\sum_{h=1}^{k} \mathtt{A}_h^{\mathrm{quadr}}(u^L, u^M, u^R) + \mathtt{A}_h^{\mathrm{canc}}(u^L, u^M, u^R) \right].$$

- *if $s_k' s_k'' < 0$, then*

$$\left.\begin{array}{l} \|(\tilde{\sigma}_k' \vartriangle \tilde{\sigma}_k'') - \tilde{\sigma}_k\|_{L^1(\boldsymbol{I}(s_k' + s_k''))} \\ \left\| \left(\dfrac{d^2 \tilde{f}_k'}{d\tau^2} \vartriangle \dfrac{d^2 \tilde{f}_k''}{d\tau^2} \right) - \dfrac{d^2 \tilde{f}_k}{d\tau^2} \right\|_{L^1(\boldsymbol{I}(s_k' + s_k''))} \end{array}\right\} \leq \mathcal{O}(1) \left[\sum_{h=1}^{k} \mathtt{A}_h^{\mathrm{quadr}}(u^L, u^M, u^R) + \mathtt{A}_h^{\mathrm{canc}}(u^L, u^M, u^R) \right].$$

PROOF. *Step 1.* First we prove that for each $k = 1, \dots, N$, it holds

$$\left| \tilde{u}_k''(s_k' + s_k'') - \tilde{u}_k(s_k' + s_k'') \right| \leq \mathcal{O}(1) \sum_{h=1}^{k} \left[\mathtt{A}_h^{\mathrm{quadr}}(u^L, u^M, u^R) + \mathtt{A}_h^{\mathrm{canc}}(u^L, u^M, u^R) \right]. \tag{3.25}$$

Recalling that $\tilde{u}_k''$ is defined on $s_k' + \mathbf{I}(s_k'')$, set

$$\widehat{\gamma}_k = (\hat{u}_k, \hat{v}_k, \hat{\sigma}_k) := \begin{cases} \gamma_k\big(\tilde{u}_k'(0), s_k' + s_k''\big) & \text{if } s_k' s_k'' \geq 0 \text{ or } \Big(s_k' s_k'' < 0 \text{ and } |s_k'| > |s_k''|\Big), \\ \gamma_k\big(\tilde{u}_k''(0), s_k' + s_k''\big) & \text{if } s_k' s_k'' < 0 \text{ and } |s_k''| > |s_k'|. \end{cases}$$

In order to prove (3.25), distinguish three cases:

- first assume that $s_k' s_k'' \geq 0$; the following computation holds:

$$\begin{aligned} \left| \tilde{u}_k''(s_k' + s_k'') - \tilde{u}_k(s_k' + s_k'') \right| &\leq \left| \tilde{u}_k''(s_k' + s_k'') - \hat{u}_k(s_k' + s_k'') \right| + \left| \hat{u}_k(s_k' + s_k'') - \tilde{u}_k(s_k' + s_k'') \right| \\ \text{(by Lemma 3.14 and Definition 3.16)} &\leq \mathcal{O}(1) \Big[\mathtt{A}_k^{\mathrm{quadr}}(u^L, u^M, u^R) + \left| \hat{u}_k(s_k' + s_k'') - \tilde{u}_k(s_k' + s_k'') \right| \Big] \\ \text{(by Lemma 3.7)} &\leq \mathcal{O}(1) \Big[\mathtt{A}_k^{\mathrm{quadr}}(u^L, u^M, u^R) + \left| \hat{u}_k(0) - \tilde{u}_k(0) \right| \Big]; \end{aligned}$$

- now assume that $s'_k s''_k < 0$ and $|s'_k| \geq |s''_k|$; in this case it holds, applying again Lemma 3.7 and using the fact that $\tilde{u}''_k(s'_k) = \tilde{u}'_k(s'_k)$,

$$\begin{aligned}
\big|\tilde{u}''_k(s'_k+s''_k) - \tilde{u}_k(s'_k+s''_k)\big| &\leq \big|\tilde{u}''_k(s'_k+s''_k) - \tilde{u}''_k(s'_k)\big| + \big|\tilde{u}'_k(s'_k) + \tilde{u}'_k(s'_k+s''_k)\big| \\
&\quad + \big|\tilde{u}'_k(s'_k+s''_k) - \hat{u}_k(s'_k+s''_k)\big| + \big|\hat{u}'_k(s'_k+s''_k) - \tilde{u}_k(s'_k+s''_k)\big| \\
&\leq \mathcal{O}(1)\Big[|s''_k| + \big|\tilde{u}'_k(0) - \tilde{u}_k(0)\big|\Big] \\
&= \mathcal{O}(1)\Big[\mathtt{A}^{\mathrm{canc}}_k(u^L,u^M,u^R) + \big|\tilde{u}'_k(0) - \tilde{u}_k(0)\big|\Big];
\end{aligned}$$

- finally assume that $s'_k s''_k < 0$ and $|s'_k| < |s''_k|$ and perform the following computation:

$$\begin{aligned}
\big|\tilde{u}''_k(s'_k+s''_k) - \tilde{u}_k(s'_k+s''_k)\big| &\leq \big|\tilde{u}''_k(s'_k+s''_k) - \hat{u}_k(s'_k+s''_k)\big| + \big|\hat{u}_k(s'_k+s''_k) - \tilde{u}_k(s'_k+s''_k)\big| \\
\text{(by Corollary 3.10 and Lemma 3.7)} &\leq \mathcal{O}(1)\Big[|s'_k| + \big|\hat{u}_k(0) - \tilde{u}_k(0)\big|\Big] \\
\text{(since } \hat{u}_k(0) = \tilde{u}''_k(0)\text{)} &\leq \mathcal{O}(1)\Big[|s'_k| + \big|\tilde{u}''_k(0) - \tilde{u}_k(0)\big|\Big] \\
\text{(since } \tilde{u}'_k(s'_k) = \tilde{u}''_k(s'_k)\text{)} &\leq \mathcal{O}(1)\Big[|s'_k| + \big|\tilde{u}''_k(0) - \tilde{u}''_k(s'_k)\big| + \big|\tilde{u}'_k(s'_k) - \tilde{u}'_k(0)\big| + \big|\tilde{u}'_k(0) - \tilde{u}_k(0)\big|\Big] \\
\text{(since } \tilde{u}'_k, \tilde{u}''_k \text{ are Lipschitz)} &\leq \mathcal{O}(1)\Big[|s'_k| + \big|\tilde{u}'_k(0) - \tilde{u}_k(0)\big|\Big] \\
&\leq \mathcal{O}(1)\Big[\mathtt{A}^{\mathrm{canc}}_k(u^L,u^M,u^R) + \big|\tilde{u}'_k(0) - \tilde{u}_k(0)\big|\Big];
\end{aligned}$$

Summarizing the three previous cases, we obtain

$$\big|\tilde{u}''_k(s'_k+s''_k) - \tilde{u}_k(s'_k+s''_k)\big| \leq \mathcal{O}(1)\Big[\mathtt{A}^{\mathrm{quadr}}_k(u^L,u^M,u^R) + \mathtt{A}^{\mathrm{canc}}_k(u^L,u^M,u^R) + \big|\tilde{u}'_k(0) - \tilde{u}_k(0)\big|\Big]. \tag{3.26}$$

If $k = 1$, $\tilde{u}'_1(0) = \tilde{u}_1(0) = u^L$, and thus (3.26) yields (3.25). If $k \geq 2$, one observes that

$$\tilde{u}'_k(0) - \tilde{u}_k(0) = \tilde{u}''_{k-1}(s'_{k-1}+s''_{k-1}) - \tilde{u}_{k-1}(s'_{k-1}+s''_{k-1})$$

and argues by induction to obtain (3.25).

Step 2. Using Step 1 we can now conclude the proof of the lemma. We will prove only the inequalities related to the σ-component, the proof of the other one being completely analogous. We again study the three cases separately:

- if $s'_k s''_k \geq 0$, it holds

$$\begin{aligned}
\big\|(\tilde{\sigma}'_k \cup \tilde{\sigma}''_k) - \tilde{\sigma}_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} &\leq \big\|(\tilde{\sigma}'_k \cup \tilde{\sigma}''_k) - \hat{\sigma}_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} + \big\|\hat{\sigma}_k - \tilde{\sigma}_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} \\
\text{(by Lemmas 3.14 and 3.7)} &\leq \mathcal{O}(1)\Big[\mathtt{A}^{\mathrm{quadr}}_k(u^L,u^M,u^R) + \big|\tilde{u}'_k(0) - \tilde{u}_k(0)\big|\Big];
\end{aligned}$$

- if $s'_k s''_k < 0$ and $|s'_k| \geq |s''_k|$, it holds

$$\begin{aligned}
\big\|(\tilde{\sigma}'_k \vartriangle \tilde{\sigma}''_k) - \tilde{\sigma}_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} &\leq \big\|(\tilde{\sigma}'_k \vartriangle \tilde{\sigma}''_k) - \hat{\sigma}_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} + \big\|\hat{\sigma}_k - \tilde{\sigma}_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} \\
\text{(by Corollary 3.10 and Lemma 3.7)} &\leq \mathcal{O}(1)\Big[\mathtt{A}^{\mathrm{canc}}_k(u^L,u^M,u^R) + \big|\tilde{u}'_k(0) - \tilde{u}_k(0)\big|\Big];
\end{aligned}$$

- if $s'_k s''_k < 0$ and $|s'_k| < |s''_k|$, it holds

$$
\begin{aligned}
\big\|(\tilde\sigma'_k \vartriangle \tilde\sigma''_k) - \tilde\sigma_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} &\leq \big\|(\tilde\sigma'_k \vartriangle \tilde\sigma''_k) - \hat\sigma_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} + \big\|\hat\sigma_k - \tilde\sigma_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} \\
\text{(by Corollary 3.10 and Lemma 3.7)} &\leq \mathcal{O}(1)\Big[\mathtt{A}_k^{\text{canc}}(u^L,u^M,u^R) + \big|\tilde u''_k(0) - \tilde u_k(0)\big|\Big] \\
\text{(since } \tilde u'_k(s'_k) = \tilde u''_k(s'_k)) &\leq \mathcal{O}(1)\Big[\mathtt{A}_k^{\text{canc}}(u^L,u^M,u^R) + \big|\tilde u''_k(0) - \tilde u''_k(s'_k)\big| \\
&\qquad + \big|\tilde u'_k(s'_k) - \tilde u'_k(0)\big| + \big|\tilde u'_k(0) - \tilde u_k(0)\big|\Big] \\
&\leq \mathcal{O}(1)\Big[\mathtt{A}_k^{\text{canc}}(u^L,u^M,u^R) + \big|\tilde u'_k(0) - \tilde u_k(0)\big|\Big].
\end{aligned}
$$

Summarizing,

$$
\left.\begin{aligned}
&\big\|(\tilde\sigma'_k \cup \tilde\sigma''_k) - \tilde\sigma_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))} \\
&\big\|(\tilde\sigma'_k \vartriangle \tilde\sigma''_k) - \tilde\sigma_k\big\|_{L^\infty(\mathbf{I}(s'_k+s''_k))}
\end{aligned}\right\} \leq \mathcal{O}(1)\Big[\mathtt{A}_k^{\text{quadr}}(u^L,u^M,u^R) + \mathtt{A}_k^{\text{canc}}(u^L,u^M,u^R) + \big|\tilde u'_k(0) - \tilde u_k(0)\big|\Big]. \tag{3.27}
$$

If $k = 1$, (3.27) together with the fact that $\tilde u'_1(0) = \tilde u_1(0) = u^L$ yields the thesis. If $k \geq 2$, one observes that $\tilde u'_k(0) = \tilde u''_{k-1}(s'_k + s''_k)$ and $\tilde u_k(0) = \tilde u_{k-1}(s'_k + s''_k)$; hence, using (3.27) and (3.25) of Step 1, one gets the statement. □

Finally, let us analyze the perturbation of the total variation due to nonlinearity.

LEMMA 3.21. *For any $k = 1, \dots, N$ it holds*

$$
\left.\begin{aligned}
&\|\tilde\sigma_k - \sigma_k\|_{L^1(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \\
&\left\|\frac{d^2\tilde f_k}{d\tau^2} - \frac{d^2 f_k}{d\tau^2}\right\|_{L^1(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))}
\end{aligned}\right\} \leq \mathcal{O}(1)\sum_{h=1}^k \big|s_h - (s'_h + s''_h)\big|.
$$

PROOF. We divide the proof in two steps.
Step 1. We prove first that for any $k = 1, \dots, N$,

$$
\|\tilde u_k - u_k\|_{L^\infty(\mathbf{I}(s'_k+s''_k)\cap\mathbf{I}(s_k))} \leq \mathcal{O}(1)\sum_{h=1}^k \big|s_h - (s'_h + s''_h)\big|. \tag{3.28}
$$

The proof is by induction on k. If $(s'_k + s''_k)s_k \leq 0$, there is nothing to prove. Hence, let us assume $(s'_k + s''_k)s_k > 0$. Set $\overline\gamma_k = (\overline u_k, \overline v_k, \overline\sigma_k) := \gamma_k\big(\tilde u_k(0), s_k\big)$. It holds

$$
\begin{aligned}
\big\|\tilde u_k - u_k\big\|_{L^\infty(I(s'_k+s''_k)\cap I(s_k))} &\leq \big\|\tilde u_k - \overline u_k\big\|_{L^\infty(I(s'_k+s''_k)\cap I(s_k))} + \big\|\overline u_k - u_k\big\|_{L^\infty(I(s'_k+s''_k)\cap I(s_k))} \\
\text{(by Corollary 3.10 and Lemma 3.7)} &\leq \mathcal{O}(1)\Big[\big||s_k| - |s'_k + s''_k|\big| + \big|\overline u_k(0) - u_k(0)\big|\Big].
\end{aligned} \tag{3.29}
$$

If $k=1$, (3.29) yields (3.28). If $k \geq 2$, observe that

$$
\begin{aligned}
\big|\overline{u}_k(0) - u_k(0)\big| &= \big|\tilde{u}_k(0) - u_k(0)\big| \\
&= \big|\tilde{u}_{k-1}(s'_{k-1} + s''_{k-1}) - u_{k-1}(s_{k-1})\big| \\
&\leq \begin{cases} \begin{aligned} &\big|\tilde{u}_{k-1}(s'_{k-1} + s''_{k-1}) - \tilde{u}_{k-1}(s_{k-1})\big| \\ &\qquad + \big|\tilde{u}_{k-1}(s_{k-1}) - u_{k-1}(s_{k-1})\big| \end{aligned} & \text{if } |s'_{k-1} + s''_{k-1}| \geq |s_{k-1}|, \\ \begin{aligned} &\big|\tilde{u}_{k-1}(s'_{k-1} + s''_{k-1}) - u_{k-1}(s'_{k-1} + s''_{k-1})\big| \\ &\qquad + \big|u_{k-1}(s'_{k-1} + s''_{k-1}) - u_{k-1}(s_{k-1})\big| \end{aligned} & \text{if } |s'_{k-1} + s''_{k-1}| < |s_{k-1}|, \end{cases} \\
&\leq \mathcal{O}(1)\Big[\big|s_{k-1} - (s'_{k-1} + s''_{k-1})\big| + \big\|\tilde{u}_{k-1} - u_{k-1}\big\|_{L^\infty(I(s'_{k-1} + s''_{k-1}) \cap I(s_{k-1}))}\Big] \\
\text{(by induction)} &\leq \sum_{h=1}^{k-1} \Big|s_h - (s'_h + s''_h)\Big|.
\end{aligned}
$$

Hence, using (3.29), we get

$$
\big\|\tilde{u}_k - u_k\big\|_{L^\infty(I(s'_k + s''_k) \cap I(s_k))} \leq \mathcal{O}(1)\Big[\big|s_k - (s'_k + s''_k)\big| + \big|\overline{u}_k(0) - u_k(0)\big|\Big] \leq \mathcal{O}(1)\sum_{h=1}^{k} \big|s_h - (s'_h + s''_h)\big|.
$$

Step 2. We conclude now the proof of the Lemma. In particular, we prove only the inequality related to the σ component, the other one being completely similar. It holds

$$
\begin{aligned}
\big\|\tilde{\sigma}_k - \sigma_k\big\|_{L^\infty(I(s'_k + s''_k) \cap I(s_k))} &\leq \big\|\tilde{\sigma}_k - \overline{\sigma}_k\big\|_{L^\infty(I(s'_k + s''_k) \cap I(s_k))} + \big\|\overline{\sigma}_k - \sigma_k\big\|_{L^\infty(I(s'_k + s''_k) \cap I(s_k))} \\
&\text{(by Corollary 3.10 and Lemma 3.7)} \\
&\leq \mathcal{O}(1)\Big[\big||s_k| - |s'_k + s''_k|\big| + \big|\overline{u}_k(0) - u_k(0)\big|\Big] \\
&= \mathcal{O}(1)\Big[\big||s_k| - |s'_k + s''_k|\big| + \big|\tilde{u}_k(0) - u_k(0)\big|\Big] \\
\text{(by (3.28))} &\leq \mathcal{O}(1)\sum_{h=1}^{k} \big|s_h - (s'_h + s''_h)\big|.
\end{aligned}
$$

□

Applying Theorem 2.9, we immediately obtain the following corollary.

COROLLARY 3.22. *For any $k = 1, \dots, N$ it holds*

$$
\left.\begin{aligned} &\|\tilde{\sigma}_k - \sigma_k\|_{L^1(\boldsymbol{I}(s'_k + s''_k) \cap \boldsymbol{I}(s_k))} \\ &\left\|\frac{d^2 \tilde{f}_k}{d\tau^2} - \frac{d^2 f_k}{d\tau^2}\right\|_{L^1(\boldsymbol{I}(s'_k + s''_k) \cap \boldsymbol{I}(s_k))} \end{aligned}\right\} \leq \mathcal{O}(1)\bigg[\mathtt{A}^{\text{trans}}(u^L, u^M, u^R) + \sum_{h=1}^{k} \mathtt{A}_h^{\text{cubic}}(u^L, u^M, u^R)\bigg].
$$

It is easy to see now that Theorem 3.18 follows from Lemmas 3.19, 3.20 and Corollary 3.22.

3.3. A wave tracing algorithm for the Glimm approximations u^ε

Let $\varepsilon > 0$ be a fixed positive number and let u^ε be the associated Glimm approximate solution. In this section we introduce an algorithm which splits the elementary wavefronts present in u^ε into infinitesimal waves and traces the position and the strength of each wave when time goes on. This sections is divided in four parts. First in Section 3.3.1 we give the definition of *wave tracing* for the Glimm approximate solution u^ε up to a fixed time $T > 0$ and we introduce some related notions, in particular the notion of *interval of waves* which will be fundamental in the following. Then, in Section 3.3.2, we prove that for any ε and for any T it is possible to construct a wave tracing for u^ε up to time T. Finally in Sections 3.3.3 and

3.3.4 we introduce, respectively, the projection maps $\Phi_k(t)$, $k = 1, \dots, N$ and the effective fluxes $\mathtt{f}_k^{\mathrm{eff}}(t)$ which will be widely used in this chapter in order to define the functional $\mathfrak{Q}$ and to prove its properties.

Some of the results of this section, in particular the existence of a wave tracing for any ε and any T satisfying some suitable properties, will be used also in Chapter 5, where we will pass to the limit, as $\varepsilon \to 0$, the family of wave tracing, in order to get a *Lagrangian representation* of the exact solution of the Cauchy problem (3.1).

All the objects we will define in this section depend on the Glimm approximation u^ε (and thus on ε) and on the fixed time T. However, since we are working at ε and T fixed, to avoid heavy notations we will omit to explicitly denote this dependence.

3.3.1. Definition of wave tracing. Given a Glimm approximate solution u^ε (see Section 2.3), a *wave tracing for u^ε up to a fixed time $T > 0$* is a $(N+3)$-tuple

$$\mathcal{E} = (L_0, \dots, L_N, \mathtt{x}, \rho)$$

(we do not explicitly denote the dependence of the objects in $\mathcal{E}$ on ε and T) where

$$\begin{aligned}
&L_0 \leq \dots \leq L_N \text{ and } (L_{k-1}, L_k] \text{ is called the } \textit{set of } k\textit{-th waves},\\
&\mathtt{x} : [0,T] \times (L_0, L_N] \to \mathbb{R} \text{ is the } \textit{position function}\\
&\rho : [0,T] \times (L_0, L_N] \to \{-1, 0, 1\}, \text{ is the } \textit{density function}
\end{aligned}$$

such that, denoting by t the time variable in $[0,T]$ and by w the "wave" variable in $(L_0, L_N]$, the following properties (1)-(5) holds:

(1) (*regularity properties of* $\mathtt{x}$ *in* w): for any fixed time $t \in [0,T]$, the map $w \mapsto \mathtt{x}(t,w)$ is piecewise constant and left continuous; when restricted to each $(L_{k-1}, L_k]$, $k = 1, \dots, N$, it is increasing; moreover at times $j\varepsilon \in [0,T]$, $j \in \mathbb{N}$, it takes values in the set $\mathbb{Z}\varepsilon$;

(2) (*regularity properties of* $\mathtt{x}$ *in* t): for any fixed wave $w \in (L_0, L_N]$, the map $t \mapsto \mathtt{x}(t,w)$ is piecewise linear and on each time interval $[j\varepsilon, (j+1)\varepsilon] \cap [0,T]$, $j \in \mathbb{N}$, its slope is either 0 or 1;

(3) (*regularity properties of* ρ *in* w): for any fixed time $t \in [0,T]$, for any fixed point $x \in \mathbb{R}$, and for any fixed family $k \in \{1, \dots, N\}$, if the interval

$$\mathtt{x}(t)^{-1}(x) \cap \big(L_{k-1}, L_k\big] := \Big\{w \in (L_0, L_N] \ \Big|\ \mathtt{x}(t,w) = x\Big\} \neq \emptyset,$$

then the restriction of ρ to the set $\mathtt{x}(t)^{-1}(x) \cap \big(L_{k-1}, L_k\big]$ is piecewise constant, left continuous and takes values either in $\{0,1\}$ or in $\{-1,0\}$; moreover, if the Riemann problem $(u(t,x-), u(t,x+))$ is solved by $u(t,x+) = T^N_{s_N} \circ \dots \circ T^1_{s_1} u(t,x-)$, then

$$\int_{\mathtt{x}(t)^{-1}(x) \cap (L_{k-1}, L_k]} \rho(t,w) dw = s_k;$$

(4) (*regularity properties of* ρ *in* t): for any fixed wave $w \in (L_0, L_N]$ there exist two times $t_1, t_2 \in \mathbb{N}\varepsilon$ (depending on w), such that

$$\text{either } \rho(t,w) = \chi_{[t_1,t_2) \cap [0,T]}(t) \text{ or } \rho(t,w) = -\chi_{[t_1,t_2 \cap [0,T])}(t)$$

If Properties (1)-(4) hold, it is possible to define three maps

$$\begin{aligned}
&\bar{u} : \Big\{(t,w) \in [0,T] \times \big(L_0, L_N\big] \ \Big|\ \rho(t,w) \neq 0\Big\} \to \mathbb{R}^n,\\
&\bar{v} : \Big\{(t,w) \in [0,T] \times \big(L_0, L_N\big] \ \Big|\ \rho(t,w) \neq 0\Big\} \to \mathbb{R}^n,\\
&\bar{\sigma} : [0,T] \times \big(L_0, L_N\big] \to [0,1]
\end{aligned}$$

as follows. Fix $(t,w) \in [0,T] \times \big(L_0, L_N\big]$. Assume that $w \in (L_{k-1}, L_k]$ and $\mathtt{x}(t,w) = x$. Assume also that the Riemann problem $(u(t,x-), u(t,x+))$ is solved by $u(t,x+) = T^N_{s_N} \circ \cdots \circ T^1_{s_1} u(t,x-)$ and denote by

$$\gamma_k : \mathbf{I}(s_k) \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \quad \gamma_k = (u_k, v_k, \sigma_k), \quad k = 1, \dots, n$$

the exact curves which solve the Riemann problem $(u(t,x-), u(t,x+))$. If $\rho(t,w) \neq 0$ we can now define

$$\begin{aligned} \bar{u}(t,w) &:= u_k\bigg(\int_{\inf \mathtt{x}(t)^{-1}(x) \cap (L_{k-1}, L_k]}^{w} \rho(t,y)dy\bigg), \\ \bar{v}(t,w) &:= v_k\bigg(\int_{\inf \mathtt{x}(t)^{-1}(x) \cap (L_{k-1}, L_k]}^{w} \rho(t,y)dy\bigg), \end{aligned} \tag{3.30}$$

and, for any $w \in (L_{k-1}, L_k]$, regardless of $\rho(t,w)$,

$$\bar{\sigma}(t,w) := \begin{cases} \sigma_k\bigg(\displaystyle\int_{\inf \mathtt{x}(t)^{-1}(x) \cap (L_{k-1}, L_k]}^{w} \rho(t,y)dy\bigg) & \text{if } \displaystyle\int_{\mathtt{x}(t)^{-1}(x) \cap (L_{k-1}, L_k]} \rho(t,y)dy \neq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.31}$$

The definitions are well posed thanks to the regularity properties of ρ in w (Point (3) above). Notice also that we do not need to write an index k in $\bar{u}, \bar{v}, \bar{\sigma}$, since for any given $w \in \mathcal{W}$ there exists a unique k such that $w \in (L_{k-1}, L_k]$.

The further property we require on $\mathcal{E}$ in order to have a wave tracing is that

(5) (*relation between* $\mathtt{x}$ *and* σ) For any $w \in (L_0, L_N]$ and for any $j \in \mathbb{N}$ such that $j\varepsilon \in [\varepsilon, T]$, if $\rho\big((j-1)\varepsilon, w\big) \neq 0$, then

$$\mathtt{x}\big(j\varepsilon, w\big) = \begin{cases} \mathtt{x}\big((j-1)\varepsilon, w\big) & \text{if } \bar{\sigma}\big((j-1)\varepsilon, w\big) \leq \vartheta_j, \\ \mathtt{x}\big((j-1)\varepsilon, w\big) + \varepsilon & \text{if } \bar{\sigma}\big((j-1)\varepsilon, w\big) > \vartheta_j. \end{cases}$$

REMARK 3.23. What we have in mind is the following. For any $k = 1, \dots, N$, the set $(L_{k-1}, L_k]$ is the *set of (infinitesimal) waves of the* k*-th family*. The map $\mathtt{x}(t,w)$ describes the *position* of a given wave w at a fixed time $t \in [0, i\varepsilon]$. The map $\rho(t,w)$ is the *density* of a given wave w at a fixed time $t \in [0, i\varepsilon]$. For fixed w, the times t_1, t_2 given by Property (4) (which depend on w) are respectively the time at which the wave w is created and the time at which w is canceled. Moreover, the value of $\rho(t,w)$ in the time interval $[t_1, t_2)$ is the sign of w: $+1$ if the infinitesimal wave w belongs to a positive wavefront, -1 if the infinitesimal wave w belongs to a negative wavefront.

Moreover, for any $(t,x) \in [0,\infty) \times \mathbb{R}$, if

$$\int_{\mathtt{x}(t)^{-1}(x) \cap (L_{k-1}, L_k]} \rho(t,y)dy \neq 0,$$

then the 3-tuple $(\bar{u}, \bar{v}, \bar{\sigma})$, restricted on the set of k-waves whose density is not zero, coincides (up to the fact that the set k-waves whose density is not zero is a finite union of intervals and not one single interval) with the curve γ_k used in the solution of the Riemann problem $(u(t,x-), u(t,x+))$. The map $\bar{\sigma}$ is then extended by continuity on the set of all k-waves located at x. On the other side, if $\int_{\mathtt{x}(t)^{-1}(x) \cap (L_{k-1}, L_k]} \rho(t,y)dy = 0$, the maps $\bar{u}, \bar{v}$ are not defined, while the map $\bar{\sigma}$ is set identically equal to 1.

It is not hard to see that:

- (*regularity properties of* $\bar{\sigma}$ *in* w) the map $w \mapsto \bar{\sigma}(t,w)$ is Lipschitz and increasing on each $\mathtt{x}(t)^{-1}(x) \cap (L_{k-1}, L_k]$;

- (*regularity properties of* $\bar{u}, \bar{v}, \bar{\sigma}$ *in* t) the maps $t \mapsto \bar{u}(t,w), \bar{v}(t,w), \bar{\sigma}(t,w)$ are piecewise constant with jumps at nodal times $j\varepsilon, j \in \mathbb{N}$.

Given a wave tracing $\mathcal{E}$ as above, we introduce the following notation:

$$\mathcal{W} := \big(L_0, L_N\big] \text{ the } \textit{set of waves},$$
$$\mathcal{W}_k := \big(L_{k-1}, L_k\big] \text{ the } \textit{set of waves of the } k\textit{-th family}$$

Define also the *creation time* and *cancellation time* of a wave $w \in \mathcal{W}$ as

$$\mathtt{t}^{\mathrm{cr}}(w) := \min\big\{t \in [0, i\varepsilon] \;\big|\; \rho(t,w) \neq 0\big\} \qquad \mathtt{t}^{\mathrm{canc}}(w) := \sup\big\{t \in [0, i\varepsilon] \;\big|\; \rho(t,w) \neq 0\big\},$$

(the definition is well posed thanks to Property (4) in the definition of wave tracing). Define also the *sign* of a wave w as

$$\mathcal{S}(w) := \operatorname{sign}\Big(\rho\big(\mathtt{t}^{\mathrm{cr}}(w), w\big)\Big).$$

Observe that $\rho(t,w) = \mathcal{S}(w)\chi_{[\mathtt{t}^{\mathrm{cr}}(w), \mathtt{t}^{\mathrm{canc}}(w))}$. For any time $t \in [0,T]$ and any point $x \in \mathbb{R}$, define also, for the sake of convenience,

$$\begin{aligned} \mathcal{W}_k(t) &:= \big\{w \in \mathcal{W}_k \;\big|\; \mathtt{t}^{\mathrm{cr}}(w) \leq t < \mathtt{t}^{\mathrm{canc}}(w)\big\}, \\ \mathcal{W}_k^{\pm}(t) &:= \big\{w \in \mathcal{W}_k(t) \;\big|\; \mathcal{S}(w) = \pm 1\big\}, \\ \mathcal{W}_k(t,x) &:= \big\{w \in \mathcal{W}_k(t) \;\big|\; \mathtt{x}(t,w) = x\big\}, \end{aligned} \tag{3.32}$$

and for any $i \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $((i+1)\varepsilon, m\varepsilon) \in [0,T] \times \mathbb{R}$:

$$\begin{aligned} \mathcal{W}_k^{(0)}(i\varepsilon, m\varepsilon) &:= \mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathtt{x}\big((i+1)\varepsilon\big)^{-1}(m\varepsilon), \\ \mathcal{W}_k^{(1)}(i\varepsilon, m\varepsilon) &:= \mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathtt{x}\big((i+1)\varepsilon\big)^{-1}\big((m+1)\varepsilon\big). \end{aligned}$$

Observe that

$$\mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k\big((i+1)\varepsilon\big) = \mathcal{W}_k^{(0)}(i\varepsilon, m\varepsilon) \cup \mathcal{W}_k^{(1)}(i\varepsilon, m\varepsilon).$$

Finally, we introduce one of the most important definition of this chapter.

Definition 3.24. Let $t \in [0,T]$ be a fixed time. Let $\mathcal{I} \subseteq \mathcal{W}_k$ be a set of k-waves. We say that $\mathcal{I}$ is an *interval of waves (i.o.w.) at time* t if $\mathcal{I} \subseteq \mathcal{W}_k^{\pm}(t)$ and $\mathcal{I}$ is an interval in the order $\big(\mathcal{W}_k^{\pm}(t), \leq\big)$, i.e. for any $w, w' \in \mathcal{I}$ and for any $y \in \mathcal{W}_k^{\pm}(t)$,

$$\text{if } w \leq y \leq w', \text{ then } y \in \mathcal{I}.$$

3.3.2. Explicit construction of a wave tracing. We now explicitly construct a wave tracing for a given Glimm approximate solution u^ε up to a fixed time T. In particular we will prove the following theorem.

Theorem 3.25. *Given a Glimm approximate solution* u^ε *and a time* $T > 0$*, there exists a wave tracing* $\mathcal{E} = (L_0, \dots, L_N, \mathtt{x}, \rho)$*, which moreover satisfies the additional conditions:*

(a) (bound on the number of waves)*:* $L_N - L_0 \leq C(F)\mathrm{Tot.Var.}(\bar{u})$*, where* C *is a constant which depends only on* F *and not on* ε*;*

(b) (the waves are created on the on the extrema) *for each* $j \in \mathbb{N}$*,* $m \in \mathbb{Z}$*,* $(j\varepsilon, m\varepsilon) \in [\varepsilon, T] \times \mathbb{R}$*, the set*

$$\mathcal{W}_k(j\varepsilon, m\varepsilon) \cap \mathcal{W}_k\big((j-1)\varepsilon\big)$$

is an interval of waves both at time $j\varepsilon$ *and at time* $(j-1)\varepsilon$*.*

As an immediate consequence of Property (b) above, we get the following corollary.

Corollary 3.26. *The following hold.*

(1) *Let $i \in \mathbb{N}$, $m \in \mathbb{Z}$, $\mathcal{I} \subseteq \mathcal{W}_k^{(1)}((i-1)\varepsilon,(m-1)\varepsilon) \cup \mathcal{W}_k^{(0)}((i-1)\varepsilon, m\varepsilon)$ be an i.o.w. at time $(i-1)\varepsilon$. Then either $\mathcal{I} \cap \mathcal{W}_k(i\varepsilon)$ is empty or it is an i.o.w. both at time $(i-1)\varepsilon$ and at time $i\varepsilon$.*

(2) *Let $\mathcal{I} \subseteq \mathcal{W}_k(i\varepsilon, m\varepsilon)$ be an i.o.w. at time $i\varepsilon$. Then either $\mathcal{I} \cap \mathcal{W}_k((i-1)\varepsilon)$ is empty or it is an i.o.w. both at time $(i-1)\varepsilon$ and at time $i\varepsilon$.*

PROOF OF THEOREM 3.25. The proof is divided in three steps:

(i) First we construct by recursion on $i \in \mathbb{N}$ a $(N+3)$-tuple $\mathcal{E}^i := (L_0^i, \dots, L_N^i, \mathtt{x}^i, \rho^i)$, where

$$L_0^i \leq \cdots \leq L_N^i,$$
$$\mathtt{x}^i : [0, i\varepsilon] \times (L_0^i, L_N^i] \to \mathbb{R},$$
$$\rho^i : [0, i\varepsilon] \times (L_0^i, L_N^i] \to \{-1, 0, 1\},$$

such that Property (3) in the definition of wave tracing (page 52) is satisfied (this property is needed for the recursive step);

(ii) then we prove by induction that, for any $i \in \mathbb{N}$, $\mathcal{E}^i$ is a wave tracing for u^ε up to time $i\varepsilon$.

(iii) finally we prove that, for any i, Properties (a), (b) above hold.

The conclusion will follow immediately choosing $\mathcal{E} := \mathcal{E}^i$, where $i := \min \{j \in \mathbb{N} \mid j\varepsilon \geq T\}$.

Step (i). We explicitly construct, by recursion on $i \in \mathbb{N}$, the $(N+3)$-tuple $\mathcal{E}$ which satisfies Property (3) in the definition of wave tracing (page 52).

Base case of the recursion. For $i = 0$, set

$$L_0^0 := 0, \qquad L_k^0 := L_{k-1}^0 + \sum_{m \in \mathbb{Z}} |s_k^{0,m}|, \text{ for } k = 1, \dots, N.$$

We define $\mathtt{x}^0(0, w)$ setting

$$\mathtt{x}^0(0, w) = m\varepsilon \text{ for any } w \in \sum_{r<m} s_k^{0,r} + \mathbf{I}\big(|s_k^{0,m}|\big).$$

and ρ^0 setting

$$\rho^0(0, w) := \begin{cases} +1 & \text{if } \mathtt{x}^0(0, w) = m\varepsilon \text{ and } s_k^{0,m} > 0, \\ -1 & \text{if } \mathtt{x}^0(0, w) = m\varepsilon \text{ and } s_k^{\varepsilon,0,m} < 0. \end{cases}$$

Property (3) is straightforward.

Recursion step. Assume now that we have defined $\mathcal{E}^{i-1}$, $i \geq 1$, satisfying Property (3) and let us define $\mathcal{E}^i$ with the same property. Let $\bar{\sigma}^{i-1}$ be the map defined in (3.31) for $\mathcal{E}^{i-1}$. The definition of $\bar{\sigma}^{i-1}$ is well posed because $\mathcal{E}^{i-1}$ satisfies Property (3).

We can now define

$$L_0^i = 0, \qquad L_k^i = L_{k-1}^i + \big(L_k^{i-1} - L_{k-1}^{i-1}\big) + \sum_{m \in \mathbb{Z}} \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, m\varepsilon).$$

Define also, for any w such that $\mathtt{x}^{i-1}\big((i-1)\varepsilon, w\big) = m\varepsilon$, the auxiliary map

$$\phi^i(w) := \begin{cases} \big(w - L_{k-1}^{i-1}\big) + \sum_{\substack{r \in \mathbb{Z} \\ r<m}} \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) & \text{if } \bar{\sigma}^{i-1}\big((i-1)\varepsilon, w\big) \leq \vartheta_i, \\ \big(w - L_{k-1}^{i-1}\big) + \sum_{\substack{r \in \mathbb{Z} \\ r\leq m}} \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) & \text{if } \bar{\sigma}^{i-1}\big((i-1)\varepsilon, w\big) > \vartheta_i, \end{cases} \tag{3.33}$$

which relates waves in $\mathcal{E}^{i-1}$ to waves in $\mathcal{E}^i$. Observe that the restriction of ϕ^i

$$\phi^i : (L_{k-1}^{i-1}, L_k^{i-1}] \to \big(L_{k-1}^i, L_k^i\big] \text{ for any } k = 1, \dots, N$$

is a strictly increasing map, piecewise affine, with slope equal to 1. Moreover, if $w \in (L^i_{k-1}, L^i_k] \setminus \phi^i\big((L^{i-1}_0, L^{i-1}_N]\big)$, then there exists a unique $a \in [L^i_{k-1}, L^i_k]$ and a unique $m \in \mathbb{Z}$ such that

$$w \in a + \mathbf{I}\Big(\mathtt{A}^{\mathrm{cr}}_k(i\varepsilon, m\varepsilon)\Big) \tag{3.34}$$

We define now the position map $\mathtt{x}^i$ at times $j\varepsilon$, $j = 0, 1, \dots, i-1, i$ as follows:

a) for $w = \phi^i(y)$ and for any $j = 0, 1, \dots, i-1$, set $\mathtt{x}^i\big(j\varepsilon, w\big) = \mathtt{x}^{i-1}\big(j\varepsilon, y\big)$;

b) for $w = \phi^i(y)$ and for $j = i$, set

$$\mathtt{x}^i\big(i\varepsilon, w\big) = \begin{cases} \mathtt{x}^{i-1}\big((i-1)\varepsilon, y\big) & \text{if } \bar\sigma^{i-1}\big((i-1)\varepsilon, y\big) \leq \vartheta_i \\ \mathtt{x}^{i-1}\big((i-1)\varepsilon, y\big) + \varepsilon & \text{if } \bar\sigma^{i-1}\big((i-1)\varepsilon, y\big) > \vartheta_i; \end{cases} \tag{3.35}$$

c) for $w \in (L^i_{k-1}, L^i_k] \setminus \phi^i\big((L^{i-1}_0, L^{i-1}_N]\big)$ and for $j = i$, assuming that w satisfies (3.34), set $\mathtt{x}^i\big(i\varepsilon, w\big) = m\varepsilon$;

d) for $w \in (L^i_{k-1}, L^i_k] \setminus \phi^i\big((L^{i-1}_0, L^{i-1}_N]\big)$ and for $j = 0, 1, \dots, i-1$, set

$$\begin{aligned} \mathtt{x}^i(j\varepsilon, w) := \max\bigg\{ & \mathtt{x}^i\big(i\varepsilon, w\big) - \Big(i\varepsilon - j\varepsilon\Big), \\ & \mathtt{x}^i\Big(j\varepsilon, \max\big\{y \in \phi^i\big(L^{i-1}_{k-1}, L^{i-1}_k]\big) \mid y \leq w\big\}\Big)\bigg\}, \end{aligned}$$

where we assume $\mathtt{x}^i\Big(j\varepsilon, \max\big\{y \in \phi^i\big(L^{i-1}_{k-1}, L^{i-1}_k]\big) \mid y \leq w\big\}\Big) = -\infty$ if the set $\big\{y \in \phi^i\big(L^{i-1}_{k-1}, L^{i-1}_k]\big) \mid y \leq w\big\}$ is empty.

Finally, we extend the definition of $\mathtt{x}^i$ to all times $[0, i\varepsilon]$ as the linear interpolation in each time interval $[(j-1)\varepsilon, j\varepsilon]$.

We define now the map ρ^i at times $j\varepsilon$, $j = 0, 1, \dots, i-1, i$ as follows:

a) for $j = 0, 1, \dots, i-1$ and $w = \phi^i(y)$ set $\rho^i(j\varepsilon, w) = \rho^{i-1}(j\varepsilon, y)$;

b) for $j = 0, 1, \dots, i-1$ and $w \in (L^i_{k-1}, L^i_k] \setminus \phi^i\big((L^{i-1}_0, L^{i-1}_N]\big)$ set $\rho^i(j\varepsilon, w) = 0$;

c) for $j = i$ and $w = \phi^i(y) \in (L^i_{k-1}, L^i_k]$, assuming $\mathtt{x}^i\big(i\varepsilon, w\big) = m\varepsilon$ and

$$\begin{aligned} u^\varepsilon\big((i-1)\varepsilon, (m-1)\varepsilon\big) &= T^N_{s'_N} \circ \dots \circ T^1_{s'_1}\ u^\varepsilon\big(i\varepsilon, (m-1)\varepsilon\big), \\ u^\varepsilon\big(i\varepsilon, m\varepsilon\big) &= T^N_{s''_N} \circ \dots \circ T^1_{s''_1}\ u^\varepsilon\big((i-1)\varepsilon, (m-1)\varepsilon\big), \end{aligned} \tag{3.36}$$

set

$$\rho^i\big(i\varepsilon, w\big) := \begin{cases} \rho^i\big((i-1)\varepsilon, w\big) & \text{if } \displaystyle\int_{\inf \mathtt{x}^i(i\varepsilon)^{-1}(m\varepsilon)}^{w} \rho^i\big((i-1)\varepsilon, y\big)dy \in \mathbf{I}(s^{i,m}_k) \cap \mathbf{I}(s'_k + s''_k), \\ 0 & \text{otherwise;} \end{cases}$$

d) for $j = i$ and $w \in (L^i_{k-1}, L^i_k] \setminus \phi^i\big((L^{i-1}_0, L^{i-1}_N]\big)$, assuming $\mathtt{x}^i\big(i\varepsilon, w\big) = m\varepsilon$, set $\rho^i\big(i\varepsilon, w\big) = \mathrm{sign}(s^{i,m}_k)$.

Finally, we extend the definition of ρ^i to all times $[0, i\varepsilon]$ as

$$\rho^i(t, w) = \rho^i(j\varepsilon, w) \text{ if } t \in [j\varepsilon, (j+1)\varepsilon).$$

Using the definition of $\bar\sigma^{i-1}$ (which is exactly the speed given by the Riemann problem solved at each grid point $\big((i-1)\varepsilon, m\varepsilon\big)$, extended also to waves w which have density $\rho^{i-1}\big((i-1)\varepsilon, w\big) = 0$), it is not difficult to prove that Property (3) (regularity properties of ρ^i in w) holds.

Observe also that, for any $(t,w) \in [0,(i-1)\varepsilon] \times (L_0^{i-1}, L_N^{i-1}]$,

$$\mathtt{x}^i\big(t,\phi^i(w)\big) = \mathtt{x}^{i-1}(t,w), \qquad \rho^i\big(t,\phi^i(w)\big) = \rho^{i-1}(t,w). \tag{3.37}$$

REMARK 3.27. Roughly speaking to pass from $\mathcal{E}^{i-1}$ to $\mathcal{E}^i$, we cut the set of waves $(L_0^{i-1}, L_N^{i-1}]$ in some "good" points and we insert in those points the waves which are created at time $i\varepsilon$. Then we extend the definition of the position map $\mathtt{x}^i$ on the time interval $[0,i\varepsilon)$ also for the waves created at time $i\varepsilon$. Finally we compute the new density of the waves in $\phi^i\big((L_0^{i-1}, L_N^{i-1}]\big)$ taking into account the possible cancellations which take place at time $i\varepsilon$.

Step (ii). We prove now, by induction on $i \in \mathbb{N}$, that $\mathcal{E}^i$ is a wave tracing for u^ε up to time $i\varepsilon$ Recall that we have already proved Property (3) for any $i \in \mathbb{N}$.
For $i = 0$, Property (1) is direct consequence of the definition of $\mathtt{x}^0(0,\cdot)$, while Properties (2), (4), (5) do not apply.
Let us assume now that $\mathcal{E}^{i-1}$ is a wave tracing up to time $(i-1)\varepsilon$ and let us prove that $\mathcal{E}^i$ is a wave tracing up to time $i\varepsilon$.

Properties (1) (regularity properties of $\mathtt{x}^i$ in w), (2) (regularity properties of $\mathtt{x}^i$ in t), (4) (regularity properties of ρ^i in t) are not difficult to prove. Before proving Property (5), we need the following two lemmas.

LEMMA 3.28. *For any $i \in \mathbb{N}$, for any $(t,x) \in [0,(i-1)\varepsilon] \times \mathbb{R}$, for any $k = 1,\dots,N$ and for any $w \in (L_{k-1}^{i-1}, L_k^{i-1}]$,*

$$\int_{\inf(\mathtt{x}^{i-1}(t))^{-1}(x) \cap (L_{k-1}^{i-1}, L_k^{i-1}])}^{w} \rho^{i-1}(t,y)dy = \int_{\inf(\mathtt{x}^{i}(t))^{-1}(x) \cap (L_{k-1}^{i}, L_k^{i}])}^{\phi^i(w)} \rho^{i}(t,y)dy.$$

PROOF. Define, for the sake of convenience, the domains of integrations

$$D^{i-1} := (\mathtt{x}^{i-1}(t))^{-1}(x) \cap (L_{k-1}^{i-1}, w], \qquad D^i := (\mathtt{x}^{i}(t))^{-1}(x) \cap (L_{k-1}^{i}, \phi^i(w)].$$

Since $\phi^i : (L_{k-1}^{i-1}, L_k^{i-1}] \to (L_{k-1}^{i}, L_k^{i}]$ is piecewise affine with slope equal to 1 and since, by the definition of ρ^i, $\rho^i(t,w) = 0$ for $\tilde{y} \notin D^i \setminus \phi^i(D^{i-1})$, we can make the change of variable $\tilde{y} = \phi^i(y)$. We thus have

$$\begin{aligned}\int_{D^{i-1}} \rho^{i-1}(t,y)dy &= \int_{\phi^i(D^{i-1})} \rho^{i-1}\big(t,(\phi^i)^{-1}(\tilde{y})\big)d\tilde{y} \\ &= \int_{\phi^i(D^{i-1})} \rho^{i}\big(t,\tilde{y}\big)d\tilde{y} \\ &= \int_{D^{i-1}} \rho^{i}(t,\tilde{y})d\tilde{y}.\end{aligned}$$

□

LEMMA 3.29 (Compatibility between different indices for $\bar{u}^i, \bar{v}^i, \bar{\sigma}^i$). *For any $i \in \mathbb{N}$, for any $t \in [0,(i-1)\varepsilon]$ and for any $w \in (L_0^i, L_N^i]$,*

$$\bar{\sigma}^i\big(t,\phi^i(w)\big) = \bar{\sigma}^{i-1}(t,w)$$

and, if $\rho^i\big(t,\phi^i(w)\big) = \rho^{i-1}(t,w) \neq 0$,

$$\bar{u}^i\big(t,\phi^i(w)\big) = \bar{u}^{i-1}(t,w), \qquad \bar{v}^i\big(t,\phi^i(w)\big) = \bar{v}^{i-1}(t,w).$$

PROOF. The proof is an easy consequence of previous lemma and the definition of $\bar{u}^i, \bar{v}^i, \bar{\sigma}^i$.
□

Using the two previous lemmas, we can now prove Property (5) in the definition of wave tracing. First of all notice that if w does not belong to the image of ϕ^i, then $\rho^i\big((j-1)\varepsilon, w\big) = 0$ and thus there is nothing to prove.

We can thus assume that $w = \phi^i(y)$ for some $y \in (L_0^{i-1}, L_N^{i-1}]$. If $j < i$, then, by inductive assumption,

$$\mathtt{x}^{i-1}(j\varepsilon, y) = \begin{cases} \mathtt{x}^{i-1}\big((j-1)\varepsilon, y\big) & \text{if } \bar\sigma^{i-1}\big((j-1)\varepsilon, y\big) \leq \vartheta_j, \\ \mathtt{x}^{i-1}\big((j-1)\varepsilon, y\big) + \varepsilon & \text{if } \bar\sigma^{i-1}\big((j-1)\varepsilon, y\big) > \vartheta_j. \end{cases}$$

The conclusion follows from (3.37) and Lemma 3.29. If $j = i$, then the conclusion follows the definition of $\mathtt{x}^i(i\varepsilon, w)$ in (3.35) and again from (3.37).

Step (iii). We finally prove, by induction on $i \in \mathbb{N}$, that Properties (a) and (b) hold.

Let us first prove Property (a). For $i = 0$, by Proposition 2.14, $L_N^0 - L_0^0 \leq C(F)\text{Tot.Var.}(\bar u)$. Assume now that Property (a) is satisfied for $\mathcal{E}^{i-1}$. By Theorem 2.15,

$$\big(L_N^i - L_0^i\big) - \big(L_N^{i-1} - L_0^{i-1}\big) \leq \sum_{k=1}^N \sum_{m\in\mathbb{Z}} \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, m\varepsilon) \leq \mathcal{O}(1)\Big[Q^{\mathrm{known}}((i-1)\varepsilon) - Q^{\mathrm{known}}(i\varepsilon)\Big]$$

and thus, since $t \mapsto Q^{\mathrm{known}}(t)$ is decreasing in time, Property (a) holds also for $\mathcal{E}^i$.

Let us prove now, by induction on $i \in \mathbb{N}$, Property (b). To avoid confusion, let us denote by $\mathcal{W}_k^i, \mathcal{W}_k^i(t)$, $\mathcal{W}_k^i(t,x)$ the objects defined in (3.32) related to the wave tracing $\mathcal{E}^i$. For $i = 0$ there is nothing to prove.

Assume now that Property (b) holds for $\mathcal{E}^{i-1}$ and let us prove it for $\mathcal{E}^i$. Let $w, w' \in \mathcal{W}_k^i(j\varepsilon, m\varepsilon) \cap \mathcal{W}_k^i((j-1)\varepsilon)$ and let $y \in \mathcal{W}_k^i$, $w \leq y \leq w'$. It is enough to prove that $\rho^i(j\varepsilon, y) = \rho^i((j-1)\varepsilon, y)$.

Notice first that since $\rho^i(j\varepsilon, w) = \rho^i(j\varepsilon, w') = \rho^i((j-1)\varepsilon, w) = \rho^i((j-1)\varepsilon, w')$, then there exists $\tilde w, \tilde w' \in \mathcal{W}_k^{i-1}$ such that $w = \phi^i(\tilde w)$ and $w' = \phi^i(\tilde w')$.

Now, if $j = 1, \dots, i-1$ and $y = \phi^i(\tilde y)$ for some $y \in \mathcal{W}_k^{i-1}$, we can use inductive assumption and (3.37) to conclude. If $j = 1, \dots, i-1$ and y does not belong to the image of ϕ^i, by definition $\rho^i(j\varepsilon, y) = \rho^i((j-1)\varepsilon, y) = 0$. If $j = i$, define $\tilde y := \tilde w + (y - w)$. By (3.33) and (3.35) it holds

$$\phi^i(\tilde y) = \big(\tilde y - L_{k-1}^{i-1}\big) + \sum_{\substack{r\in\mathbb{Z} \\ r<m}} \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) = \big(\tilde w + (y-w) - L_{k-1}^{i-1}\big) + \sum_{\substack{r\in\mathbb{Z} \\ r<m}} \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) = y,$$

and thus y belongs to the image of ϕ^i. Moreover, since $\rho^i(i\varepsilon, w), \rho^i(i\varepsilon, w') \neq 0$, using the same notations as in (3.36),

$$\int_{\inf \mathtt{x}^i(i\varepsilon)^{-1}(m\varepsilon)}^{w} \rho^i\big((i-1)\varepsilon, y\big)dy \in \mathbf{I}(s_k^{i,m}) \cap \mathbf{I}(s_k' + s_k'')$$

and similarly

$$\int_{\inf \mathtt{x}^i(i\varepsilon)^{-1}(m\varepsilon)}^{w'} \rho^i\big((i-1)\varepsilon, y\big)dy \in \mathbf{I}(s_k^{i,m}) \cap \mathbf{I}(s_k' + s_k'').$$

By the regularity properties of ρ^i in w, $\rho^i\big((i-1)\varepsilon, \cdot\big)$ on the set $\mathtt{x}^i(i\varepsilon)^{-1}(m\varepsilon)$ changes its sign at most once; therefore we have that

$$\int_{\inf \mathtt{x}^i(i\varepsilon)^{-1}(m\varepsilon)}^{y} \rho^i\big((i-1)\varepsilon, y\big)dy \in \mathbf{I}(s_k^{i,m}) \cap \mathbf{I}(s_k' + s_k'')$$

and thus, by definition of ρ^i, $\rho^i(i\varepsilon, y) = \rho^i\big((i-1)\varepsilon, y\big)$, which concludes the proof of Property (b). □

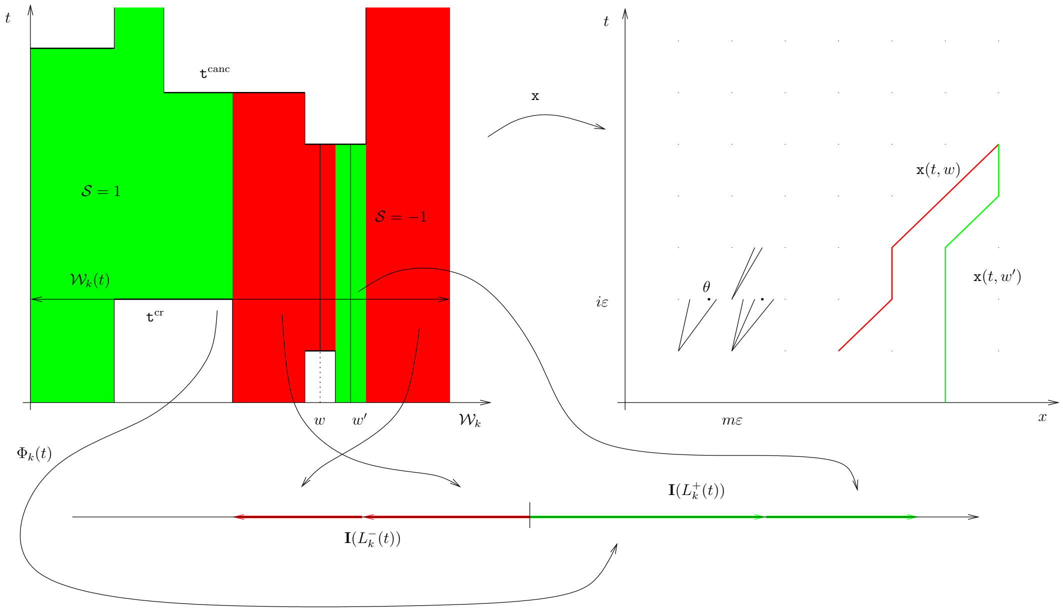

FIGURE 3. The Lagrangian representation: at each t the sum of the length of the red/green regions gives the set L_k^-, L_k^+, and $\mathtt{x}$ follows the trajectory of each wave w. The map $\Phi_k(t)$ is order reversing on $\mathcal{W}_k^-(t)$ (red) and order preserving on $\mathcal{W}_k^+(t)$ (green).

3.3.3. The projection map $\Phi_k(t)$. Let u^ε be a Glimm approximate solution with grid size ε and let $T > 0$ be a fixed time. Let $\mathcal{E} := (L_0, \dots, L_N, \mathtt{x}, \rho)$ be the wave tracing for u^ε up to time T provided by Theorem 3.25. In this and the next section we define further objects and properties related to $\mathcal{E}$, which will be widely used in the rest of this chapter.

We observe that each of the sets $\mathcal{W}_k^\pm(t)$ is in general a disjoint union of many intervals of the form $(a,b]$. It is thus convenient to introduce the following map $\Phi_k(t)$ (see Figure 3) which will be a measure-preserving and order-preserving bijection from $\mathcal{W}_k^\pm(t)$ into $\mathbf{I}\big(V_k^\pm(t)\big)$. Define thus $\Phi_k(t) : \mathcal{W}_k(t) \to \mathbf{I}\big(V_k^-(t)\big) \cup \mathbf{I}\big(V_k^+(t)\big)$ as

$$\Phi_k(t)(w) := \begin{cases} \displaystyle\int_0^w \big[\rho(t,w)\big]^+ dw \text{ if } \mathcal{S}(w) = +1, \\[2ex] \displaystyle\int_0^w -\big[\rho(t,w)\big]^- dw \text{ if } \mathcal{S}(w) = -1. \end{cases}$$

LEMMA 3.30. *If $\mathcal{A} \subseteq \mathcal{W}_k^\pm(t)$ is an interval as a subset of $\mathbb{R}$, then the restriction $\Phi_k|_{\mathcal{A}} : \mathcal{A} \to \mathbb{R}$ is an affine map with slope ± 1.*

PROOF. Assume $\mathcal{A} \subseteq \mathcal{W}_k^+(t)$, the other case being completely similar. Let $w, w' \in \mathcal{A}$. Since $\mathcal{A}$ is an interval as a subset of $\mathbb{R}$, for any $y \in \mathcal{W}$, if $w \leq y \leq w'$, then $y \in \mathcal{A}$. Moreover since $\mathcal{A} \subseteq \mathcal{W}_k^+(t)$, then $\rho(t,y) = 1$. Therefore, for any $y \in \mathcal{W}$, $w \leq y \leq w'$, it holds $\rho(t,y) = 1$. Hence

$$\Phi_k(w') - \Phi_k(w) = \int_w^{w'} \big[\rho(t,y)\big]^+ dy = \int_w^{w'} 1 dy = w' - w,$$

which proves that the restriction $\Phi_k|_{\mathcal{A}}$ is an affine map with slope equal to 1. The negative case is completely similar. $\square$

As a consequence, we get the following proposition.

PROPOSITION 3.31 (Properties of $\Phi_k(t)$). *The restrictions of $\Phi_k(t)$ to the sets $\mathcal{W}_k^{\pm}(t)$ share the following properties:*

(1) $\Phi_k(t)|_{\mathcal{W}_k^+(t)} : \mathcal{W}_k^+(t) \to \boldsymbol{I}\big(V_k^+(t)\big)$ and $\Phi_k(t)|_{\mathcal{W}_k^-(t)} : \mathcal{W}_k^-(t) \to \boldsymbol{I}\big(V_k^-(t)\big)$ are bijection;

(2) $\Phi_k(t)|_{\mathcal{W}_k^{\pm}(t)}$ is measure-preserving, i.e.

$$\Big(\Phi_k|_{\mathcal{W}_k^+(t)}\Big)_\sharp \mathcal{L}^1|_{\mathcal{W}_k^+(t)} = \mathcal{L}^1|_{\boldsymbol{I}(V_k^+(t))}, \qquad \Big(\Phi_k|_{\mathcal{W}_k^-(t)}\Big)_\sharp \mathcal{L}^1|_{\mathcal{W}_k^-(t)} = \mathcal{L}^1|_{\boldsymbol{I}(V_k^-(t))};$$

(3) $\Phi_k(t)|_{\mathcal{W}_k^+(t)}$ (resp. $\Phi_k(t)|_{\mathcal{W}_k^-(t)}$) is order-preserving (resp. reversing), i.e.

if $w, w' \in \mathcal{W}_k^+(t)$ (resp. $w, w' \in \mathcal{W}_k^-(t)$), then $\Phi_k(w) < \Phi_k(w')$ (resp. $\Phi_k(w) > \Phi_k(w')$).

PROOF. The proof is an immediate consequence of the previous lemma and of the fact that $\mathcal{W}_k^{\pm}(t)$ is an union of a finite number of intervals. □

We state now two propositions which describes the behavior of $\Phi_k(t)$ when restricted to an interval of waves.

PROPOSITION 3.32. *Let $\mathcal{I} \subseteq \mathcal{W}_k$. Then*

$$\mathcal{I} \text{ is an i.o.w. at time } t \iff \Phi_k(t)(\mathcal{I}) \text{ is an interval as a subset of } \mathbb{R}.$$

Moreover, in this case, $\mathcal{L}^1(\mathcal{I}) = \mathcal{L}^1\big(\Phi_k(t)(\mathcal{I})\big)$.

The proof is an easy consequence of Proposition 3.31 and thus it is omitted.

PROPOSITION 3.33. *Let $t, t' \in [0, T]$. Assume that $\mathcal{I} \subseteq \mathcal{W}_k^{\pm}(t) \cap \mathcal{W}_k^{\pm}(t')$ is an interval of waves both at time t and at time t'. Then*

$$\Phi_k(t') \circ \Phi_k(t)^{-1}|_{\Phi_k(t)(\mathcal{I})} : \Phi_k(t)(\mathcal{I}) \to \Phi_k(t')(\mathcal{I})$$

is an affine map with slope equal to 1.

PROOF. We assume that $\mathcal{I} \subseteq \mathcal{W}_k^+(t) \cap \mathcal{W}_k^+(t')$, the negative case being completely similar. Let $\tau, \tau' \in \Phi_k(t)(\mathcal{I})$, $\tau < \tau'$. Let $w := \Phi_k(t)^{-1}(\tau)$, $w' := \Phi_k(t)^{-1}(\tau')$. We first prove that for any $y \in [w, w']$, $[\rho(t, y)]^+ = [\rho(t', y)]^+$. Assume that $[\rho(t, y)]^+ = \rho(t, y) = 1$. Hence $y \in \mathcal{W}_k^+(t)$ and thus $y \in \mathcal{I}$, since $\mathcal{I}$ is an i.o.w. at time t. Moreover, since $\mathcal{I} \subseteq \mathcal{W}_k(t')$, then $\rho(t', y) = 1$ and thus $[\rho(t, y)]^+ = 1$. In a similar way if $[\rho(t', y)]^+ = 1$ then $[\rho(t, y)]^+ = 1$ and thus

$$[\rho(t, y)]^+ = 1 \iff [\rho(t', y)]^+ = 1. \tag{3.38}$$

This, together with the fact that $[\rho(t, y)]^+, [\rho(t', y)]^+ \in \{0, 1\}$ implies that for any $y \in [w, w']$, $[\rho(t, y)]^+ = [\rho(t', y)]^+$. Therefore

$$\begin{aligned} \Phi_k(t')\Big(\Phi_k(t)^{-1}(\tau')\Big) - \Phi_k(t')\Big(\Phi_k(t)^{-1}(\tau)\Big) &= \Phi_k(t')(w') - \Phi_k(t')(w) \\ &= \int_w^{w'} \big[\rho(t', y)\big]^+ dy \\ \text{(by (3.38))} \quad &= \int_w^{w'} \big[\rho(t, y)\big]^+ dy \\ &= \tau' - \tau \end{aligned}$$

and thus $\Phi_k(t') \circ \Phi_k(t)^{-1}|_{\Phi_k(t)(\mathcal{I})}$ is an affine map with slope equal to 1. □

We conclude this section by introducing some useful notions that we will frequently use hereinafter.

DEFINITION 3.34. Fix $\bar{t} \in [0,T]$. Let $\mathcal{I} \subseteq \mathcal{W}_k(\bar{t})$ be an interval of waves at time $\bar{t}$. Set $I := \Phi_k(\bar{t})(\mathcal{I})$. By Proposition 3.32, I is an interval in $\mathbb{R}$ (possibly made by a single point). Let us define:

- the *Rankine-Hugoniot speed given to the interval of waves* $\mathcal{I}$ *by a function* $g : \mathbb{R} \to \mathbb{R}$ as
$$\sigma^{\mathrm{rh}}(g,\mathcal{I}) := \begin{cases} \frac{g(\sup I) - g(\inf I)}{\sup I - \inf I} & \text{if } I \text{ is not a singleton,} \\ g'(I) & \text{if } I \text{ is a singleton;} \end{cases}$$
- for any $w \in \mathcal{I}$, the *entropic speed given to the wave* w *by the Riemann problem* $\mathcal{I}$ *and the flux function* g as
$$\sigma^{\mathrm{ent}}(g,\mathcal{I},w) := \begin{cases} \dfrac{d}{d\tau} \operatorname*{conv}_I g\Big(\Phi_k(\bar{t})(w)\Big) & \text{if } \mathcal{S}_k(w) = +1, \\ \dfrac{d}{d\tau} \operatorname*{conc}_I g\Big(\Phi_k(\bar{t})(w)\Big) & \text{if } \mathcal{S}_k(w) = -1. \end{cases}$$

If $\sigma^{\mathrm{rh}}(g,\mathcal{I}) = \sigma^{\mathrm{ent}}(g,\mathcal{I},w)$ for any $w \in \mathcal{I}$, we will say that $\mathcal{I}$ is *entropic* w.r.t. the function g.

We will also say that *the Riemann problem* $\mathcal{I}$ *with flux function* g *divides* w, w' if $\sigma^{\mathrm{ent}}(g,\mathcal{I},w) \neq \sigma^{\mathrm{ent}}(g,\mathcal{I},w')$.

We recall that by definition an interval of waves is made of waves with the same sign.

REMARK 3.35. Notice that σ^{ent} is always increasing on $\mathcal{I}$, whatever the sign of $\mathcal{I}$ is, by the monotonicity properties of the derivatives of the convex/concave envelopes.

REMARK 3.36. Given a function g and an interval of waves $\mathcal{I}$, we can always partition $\mathcal{I}$ through the equivalence relation
$$z \sim z' \quad \Longleftrightarrow \quad z, z' \text{ are not divided by the Riemann problem } \mathcal{I} \text{ with flux function } g.$$
As a consequence of Remark 3.35, we have that each element of this partition is an entropic interval of waves and the relation induced by the order $\leq$ on the partition (see Section 1.1) is still a total order.

3.3.4. The effective flux. We conclude this section by introducing the notion of *effective flux* $\mathtt{f}_k^{\mathrm{eff}}(t)$ *of the* k*-th family at time* t.

DEFINITION 3.37. For each family $k = 1, \dots n$ and for any fixed time $t \in [0,T]$ define the *effective flux of the* k*-th family at time* t
$$\mathtt{f}_k^{\mathrm{eff}}(t) : [V_k^-(t), V_k^+(t)] \to \mathbb{R}$$
as follows. We distinguish two cases.

- Assume first that $t \in \big[i\varepsilon, (i+1/2)\varepsilon\big)$. Notice that, by the property of $\Phi_k(t)$,
$$[V_k^-(t), V_k^+(t)] \setminus \{0\} = \bigcup_{m \in \mathbb{Z}} \Phi_k(t)\Big(\mathcal{W}_k(i\varepsilon, m\varepsilon)\Big).$$
We can thus define $\mathtt{f}_k^{\mathrm{eff}}(t)$ separately on each $\Phi_k(t)\big(\mathcal{W}_k(i\varepsilon, m\varepsilon)\big)$. Notice that for any $m \in \mathbb{Z}$, there exists a unique $a \in \mathbb{R}$ such that
$$\Phi_k(t)\big(\mathcal{W}_k(i\varepsilon, m\varepsilon)\big) = a + \mathbf{I}(s_k^{i,m}).$$
Now, assuming that $f_k^{i,m}$ is defined on $a + \mathbf{I}(s_k^{i,m})$ instead of $\mathbf{I}(s_k^{i,m})$, we define $\mathtt{f}_k^{\mathrm{eff}}(t)$ as any function whose second derivative satisfies the relation
$$D^2 \mathtt{f}_k^{\mathrm{eff}}(t)(\tau) = D^2 f_k^{i,m}(\tau).$$

- Assume now that $t \in \big[(i+1/2)\varepsilon, (i+1)\varepsilon\big)$. As before, observe that, by the property of $\Phi_k(t)$,

$$[V_k^-(t), V_k^+(t)] \setminus \{0\} = \bigcup_{m \in \mathbb{Z}} \Phi_k(t)\Big(\mathcal{W}_k^{(1)}\big(i\varepsilon, (m-1)\varepsilon\big)\Big) \cup \Phi_k(t)\Big(\mathcal{W}_k^{(0)}\big(i\varepsilon, m\varepsilon\big)\Big).$$

We can thus define $\mathtt{f}_k^{\mathrm{eff}}(t)$ separately on each $\Phi_k(t)\big(\mathcal{W}_k^{(1)}(i\varepsilon,(m-1)\varepsilon)\big)\cup\Phi_k(t)\big(\mathcal{W}_k^{(0)}(i\varepsilon, m\varepsilon)\big)$, for any fixed $m \in \mathbb{Z}$. Assume that the interaction taking place at point $\big((i+1)\varepsilon, m\varepsilon\big)$ is described by

$$u^\varepsilon\big(i\varepsilon,(m-1)\varepsilon\big) = T^N_{s'_N} \circ \cdots \circ T^1_{s'_1}\ u^\varepsilon\big((i+1)\varepsilon,(m-1)\varepsilon\big),$$

$$u^\varepsilon\big((i+1)\varepsilon, m\varepsilon\big) = T^N_{s''_N} \circ \cdots \circ T^1_{s''_1}\ u^\varepsilon\big(i\varepsilon,(m-1)\varepsilon\big).$$

Observe that there exist unique $a, b \in \mathbb{R}$ such that

$$\Phi_k(t)\Big(\mathcal{W}_k^{(1)}\big(i\varepsilon,(m-1)\varepsilon\big)\Big) = a + \mathbf{I}(s'_k), \quad \Phi_k(t)\Big(\mathcal{W}_k^{(0)}\big(i\varepsilon, m\varepsilon\big)\Big) = b + \mathbf{I}(s''_k).$$

Denote by $\tilde{f}'_k$, $\tilde{f}''_k$ the reduced fluxes associated to the two interacting Riemann problem *after the transversal interactions* (see Section 3.2). Now, assuming that $\tilde{f}'_k$, $\tilde{f}''_k$ are defined on $a + \mathbf{I}(s'_k)$ and $b + \mathbf{I}(s''_k)$ respectively (instead of $\mathbf{I}(s'_k)$ and $\mathbf{I}(s''_k)$ respectively), we define $\mathtt{f}_k^{\mathrm{eff}}(t)$ as any function whose second derivative satisfies the relation

$$D^2\mathtt{f}_k^{\mathrm{eff}}(t)(\tau) = \begin{cases} D^2\tilde{f}'_k(\tau) & \text{for } \tau \in a + \mathbf{I}(s'_k), \\ D^2\tilde{f}''_k(\tau) & \text{for } \tau \in b + \mathbf{I}(s''_k). \end{cases}$$

Remark 3.38. Let us observe the following:

(1) $\mathtt{f}_k^{\mathrm{eff}}(t,\cdot)$ is defined up to affine function;
(2) since the second derivative of $\mathtt{f}_k^{\mathrm{eff}}(t,\cdot)$ is an L^∞-function, it turns out that $\mathtt{f}_k^{\mathrm{eff}}(t,\cdot)$ is a $C^{1,1}$-function;
(3) $\mathtt{f}_k^{\mathrm{eff}}(t,\cdot) = \mathtt{f}_k^{\mathrm{eff}}(i\varepsilon,\cdot)$ for any $t \in \big[i\varepsilon,(i+1/2)\varepsilon\big)$ and $\mathtt{f}_k^{\mathrm{eff}}(t,\cdot) = \mathtt{f}_k^{\mathrm{eff}}\big((i+1/2)\varepsilon,\cdot\big)$ for any $t \in \big[(i+2)\varepsilon,(i+1)\varepsilon\big)$;

Remark 3.39. For times $t \in \big[i\varepsilon,(i+1/2)\varepsilon\big)$, the effective flux $\mathtt{f}_k^{\mathrm{eff}}(i\varepsilon)$ coincides, up to affine function, with the k-th reduced flux associated to the various Riemann problems solved at points $(i\varepsilon, m\varepsilon)$;

For times $t \in \big[(i+1/2)\varepsilon,(i+1)\varepsilon\big)$, the effective flux $\mathtt{f}_k^{\mathrm{eff}}(i\varepsilon)$ coincides, up to affine function, with the k-th reduced flux associated to the various Riemann problems which collide at $\big((i+1)\varepsilon, m\varepsilon\big)$, after the transversal interaction, but before all the non-transversal interaction;

By definition of Glimm scheme, all the interactions (transversal and non-transversal) take place at times $i\varepsilon, i \in \mathbb{N}$; the choice of splitting the intervals $[i\varepsilon,(i+1)\varepsilon)$ in the two subintervals $\big[i\varepsilon,(i+1/2)\varepsilon\big)$ and $\big[(i+1/2)\varepsilon,(i+1)\varepsilon\big)$ is due to the fact that we will need to analyze the transversal interaction separately from all the other non-transversal interactions; therefore, roughly speaking, we define the effective flux, as if all the transversal interactions take place at times $(i+1/2)\varepsilon$, $i \in \mathbb{N}$ and all the non-transversal interactions take place at times $i\varepsilon$, $i \in \mathbb{N}$.

3.4. Analysis of wave collision

Starting with this section we enter in the heart of our construction. We introduce in fact the notion of *pair of waves* (w, w') *which have already interacted* and *pair of waves* (w, w') *which have never interacted* at time $\bar{t}$. For any pair of waves (w, w') and for any fixed times

$t_1 \leq t_2$, we define an interval of waves $\mathcal{I}(t_1, t_2, w, w')$ and a partition $\mathcal{P}(t_1, t_2, w, w')$ of this interval: these objects in some sense summarize the past "common" history of the two waves, from the time of last splitting before t_1 (or from the last time in which one of them is created) up to the time t_2.

The interval $\mathcal{I}(t_1, t_2, w, w')$ and its partition $\mathcal{P}(t_1, t_2, w, w')$ will play a crucial role in the definition of the functional $\mathcal{Q}_k$ in Section 3.5 and to prove that it satisfies the inequality (3.5).

3.4.1. Wave packets. We start by defining an equivalence relation between waves, which will be useful to pass from the uncountable sets of waves $\mathcal{W}(t)$ at time t to the finite quotient set, whose elements will be called *wave packets*.

For any $\bar{t} \geq 0$ and $w \in \mathcal{W}_k(\bar{t})$, $\bar{t} \in [i\varepsilon, (i+1)\varepsilon)$, define the *wave packet to which w belongs* as the set

$$\mathcal{E}(\bar{t}, w) := \Big\{ w' \in \mathcal{W}_k(\bar{t}) \;\Big|\; \mathtt{t}^{\mathrm{cr}}(w) = \mathtt{t}^{\mathrm{cr}}(w'),\ \mathtt{x}(t, w) = \mathtt{x}(t, w') \text{ for all } t \in \big[\mathtt{t}^{\mathrm{cr}}(w), (i+1)\varepsilon\big) \Big\}. \tag{3.39}$$

In Section 3.5.5 we will denote this equivalence relation as $\bowtie$.

REMARK 3.40. Notice that is it natural to require that the condition in (3.39) holds on the time interval $[\mathtt{t}^{\mathrm{cr}}(w), (i+1)\varepsilon)$ instead of $[\mathtt{t}^{\mathrm{cr}}(w), i\varepsilon]$ since it could happen that $\mathtt{x}(i\varepsilon, w) = \mathtt{x}(i\varepsilon, w')$, but $\mathtt{x}(t, w) \neq \mathtt{x}(t, w')$ for $t > i\varepsilon$, while we want to give definitions which are "left-continuous in time".

LEMMA 3.41. *The collection $\big\{\mathcal{E}(\bar{t}, w) \;\big|\; w \in \mathcal{W}(\bar{t})\big\}$ is a finite partition of $\mathcal{W}(\bar{t})$ and the order induced by the $\leq$ is a total order both on the set $\big\{\mathcal{E}(\bar{t}, w) \;\big|\; w \in \mathcal{W}_k^+(\bar{t})\big\}$ and on the set $\big\{\mathcal{E}(\bar{t}, w) \;\big|\; w \in \mathcal{W}_k^-(\bar{t})\big\}$, $k = 1, \dots, n$.*

PROOF. Clearly $\big\{\mathcal{E}(\bar{t}, w) \;\big|\; w \in \mathcal{W}(\bar{t})\big\}$ is a partition of $\mathcal{W}(\bar{t})$. To see that it is finite, just observe that the curve $\mathtt{x}(t, \cdot)$ is uniquely determined by assigning the points $m\varepsilon = \mathtt{x}(i\varepsilon, \cdot)$, and for all fixed time $\bar{t}$ the set of nodal points supporting $D_x u^\varepsilon(t)$, $t \leq \bar{t}$, is finite. Finally, the monotonicity of $\mathtt{x}(\bar{t}, \cdot)$ implies the statement about the order. □

3.4.2. Characteristic interval. We now define the notion of pairs of waves which *have never interacted before a fixed time $\bar{t}$* and pairs of waves which *have already interacted at a fixed time $\bar{t}$* and to any pair of waves (w, w') and any pair of times $t_1 < t_2$ we will associate an interval of waves $\mathcal{I}(t_1, t_2, w, w')$.

DEFINITION 3.42. Let $\bar{t}$ be a fixed time and let $w, w' \in \mathcal{W}_k(\bar{t})$. We say that

- *w, w' interact at time $\bar{t}$* if $\mathtt{x}(\bar{t}, w) = \mathtt{x}(\bar{t}, w')$;
- *w, w' have already interacted at time $\bar{t}$* if there is $t \leq \bar{t}$ such that w, w' interact at time t;
- *w, w' have never interacted at time $\bar{t}$* if for any $t \leq \bar{t}$, they do not interact at time t.
- *w, w' will interact after time $\bar{t}$* if there is $t > \bar{t}$ such that w, w' interact at time t.
- *w, w' are joined in the real solution at time $\bar{t}$* if there is a right neighborhood of $\bar{t}$, say $[\bar{t}, \bar{t} + \zeta)$, such that they interact at any time $t \in [\bar{t}, \bar{t} + \zeta)$;
- *w, w' are divided in the real solution at time $\bar{t}$* if they are not joined at time $\bar{t}$.

LEMMA 3.43. *Assume that the waves w, w' have already interacted at time $\bar{t}$. Then they have the same sign.*

PROOF. If w, w' have already interacted at time $\bar{t}$, then there exists a point (t, x) such that $w, w' \in \mathcal{W}_k(t)$ and $\mathtt{x}(t, w) = \mathtt{x}(t, w') = x$. Since $w, w' \in \mathcal{W}_k(t)$, then, by definition of $\mathcal{W}_k(t)$, $\rho(t, w), \rho(t, w') \neq 0$. Moreover, since at time t they have the same position, by the *regularity property of* ρ (see page 52), it must be $\rho(t,w) = \rho(t,w') \neq 0$ and thus w, w' have the same sign. $\square$

REMARK 3.44. It $\bar{t} \neq i\varepsilon$ for each $i \in \mathbb{N}$, then two waves are divided in the real solution if and only if they have different position. If $\bar{t} = i\varepsilon$, they are divided if there exists a time $t > \bar{t}$, arbitrarily close to $\bar{t}$, such that w, w' have different positions at time t.

DEFINITION 3.45. Fix a time $\bar{t}$ and two k-waves $w, w' \in \mathcal{W}_k(\bar{t})$, $w < w'$. Assume that w, w' are divided in the real solution at time $\bar{t}$. Define the *time of last splitting* $\mathtt{t}^{\mathrm{split}}(\bar{t}, w, w')$ (if w, w' have already interacted at time $\bar{t}$) and the *time of next interaction* $\mathtt{t}^{\mathrm{int}}(\bar{t}, w, w')$ (if w, w' will interact after time $\bar{t}$) by the formulas

$$\mathtt{t}^{\mathrm{split}}(\bar{t}, w, w') := \max\big\{t \leq \bar{t} \mid \mathtt{x}(t,w) = \mathtt{x}(t,w')\big\},$$
$$\mathtt{t}^{\mathrm{int}}(\bar{t}, w, w') := \min\big\{t > \bar{t} \mid \mathtt{x}(t,w) = \mathtt{x}(t,w')\big\}.$$

(In the case one of sets is empty we assume the corresponding time to be $\pm\infty$.)

Observe that $\mathtt{t}^{\mathrm{split}}(\bar{t}, w, w'), \mathtt{t}^{\mathrm{int}}(\bar{t}, w, w') \in \mathbb{N}\varepsilon$.
Given two k-waves $w, w' \in \mathcal{W}_k$ and given a time $t \in [0, +\infty)$, we define the property $\mathtt{p}(t_1, w, w')$:

$$\mathtt{p}(t,w,w'): \quad \begin{array}{c} \text{“}w, w' \text{ have the same sign and moreover} \\ \text{either } w, w' \in \mathcal{W}_k(t) \text{ and they are divided at time } t \text{ in the real solution} \\ \text{or at least one between } w, w' \text{ does not belong to } \mathcal{W}_k(t)\text{”.} \end{array}$$

DEFINITION 3.46. Let $t_1 \leq t_2$, be two times. Let $w, w' \in \mathcal{W}_k(t_2)$ be two k-waves. Assume that they satisfy $\mathtt{p}(t_1, w, w')$. We define the *characteristic interval* $\mathcal{I}(t_1, t_2, w, w')$ *of* w, w' *at time* t_2 *starting from time* t_1 as follows. Assume first that $t_2 = i\varepsilon$ for some $i \in \mathbb{N}$.

(1) If at least one between w, w' does not belong to $\mathcal{W}_k(t_1)$ or $w, w' \in \mathcal{W}_k(t_1)$, but they have never interacted at time t_1, then

$$\mathcal{I}(t_1, t_2, w, w') := \begin{cases} \big\{z \in \mathcal{W}_k(t_2) \;\big|\; \mathcal{S}(z) = \mathcal{S}(w) \text{ and } z < \mathcal{E}(t_2, w')\big\} \cup \mathcal{E}(t_2, w') & \text{if } \mathtt{t}^{\mathrm{cr}}(w) \leq \mathtt{t}^{\mathrm{cr}}(w'), \\ \mathcal{E}(t_2, w) \cup \big\{z \in \mathcal{W}_k(t_2) \;\big|\; \mathcal{S}(z) = \mathcal{S}(w) \text{ and } z > \mathcal{E}(t_2, w)\big\} & \text{if } \mathtt{t}^{\mathrm{cr}}(w) > \mathtt{t}^{\mathrm{cr}}(w'); \end{cases} \tag{3.41}$$

(2) If $w, w' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1, we have to distinguish two cases:
   (a) if $t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then argue by recursion:
   - if $t_2 = t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, set
   $$\mathcal{I}(t_1, t_2, w, w') := \mathcal{W}(t_1, \mathtt{x}(t_1, w)) = \mathcal{W}(t_1, \mathtt{x}(t_1, w'));$$
   - if $t_2 = i\varepsilon > (i-1)\varepsilon \geq t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, define $\mathcal{I}(t_1, t_2, w, w')$ as the smallest interval in $(\mathcal{W}_k^{\pm}(t_2), \leq)$ which contains $\mathcal{I}(t_1, (i-1)\varepsilon, w, w') \cap \mathcal{W}_k(t_2)$, i.e.
   $$\mathcal{I}(t_1, t_2, w, w') := \Big\{z \in \mathcal{W}_k(t_2) \;\Big|\; \mathcal{S}(z) = \mathcal{S}(w) = \mathcal{S}(w') \text{ and } \exists\, y, y' \in \mathcal{I}(t_1, (i-1)\varepsilon, w, w') \cap \mathcal{W}_k(t_2) \text{ such that } y \leq z \leq y'\Big\}.$$

(b) if $t_1 > \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, set

$$\mathcal{I}(t_1, t_2, w, w') = \mathcal{I}(\mathtt{t}^{\mathrm{split}}(t_1, w, w'), t_2, w, w').$$

Finally set

$$\mathcal{I}(t_1, t_2, w, w') := \mathcal{I}(t_1, i\varepsilon, w, w') \qquad \text{for } t_2 \in [i\varepsilon, (i+1)\varepsilon).$$

REMARK 3.47. It is immediate from the definition that $\mathcal{I}(t_1, t_2, w, w')$ is an interval of waves at time t_2.

LEMMA 3.48. *Let $t_1 \leq t_2$, be two times. Let $w, w' \in \mathcal{W}_k(t_2)$ be two k-waves. Assume that they have the same sign, they have already interacted and they are divided at time t_1 (and thus they satisfy $\mathtt{p}(t_1, w, w')$). Then for any $i\varepsilon > (i-1)\varepsilon \geq t_1$,*

$$\mathcal{I}(t_1, i\varepsilon, w, w') \cap \mathcal{W}_k((i-1)\varepsilon) = \mathcal{I}(t_1, (i-1)\varepsilon, w, w') \cap \mathcal{W}_k(i\varepsilon);$$

PROOF. W.l.o.g. we can assume $t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$. The inclusion "$\supseteq$" is straightforward. To prove the inclusion "$\subseteq$", take $z \in \mathcal{I}(t_1, i\varepsilon, w, w') \cap \mathcal{W}_k((i-1)\varepsilon)$. By definition of $\mathcal{I}(t_1, i\varepsilon, w, w')$, there are $y, y' \in \mathcal{I}(t_1, (i-1)\varepsilon, w, w') \cap \mathcal{W}_k(i\varepsilon)$ such that $y \leq z \leq y'$. Since $z \in \mathcal{W}_k((i-1)\varepsilon)$ and, by Remark (3.47), $\mathcal{I}(t_1, (i-1)\varepsilon, w, w')$ is an interval of waves at time $(i-1)\varepsilon$, it must be $z \in \mathcal{I}((i-1)\varepsilon, w, w')$.

$\square$

3.4.3. Partition of the characteristic interval. Let $w, w' \in \mathcal{W}_k(t_2)$ be two k-waves. Assume that they satisfy $\mathtt{p}(t_1, w, w')$. We define a partition $\mathcal{P}(t_1, t_2, w, w')$ of the interval of waves $\mathcal{I}(t_1, t_2, w, w')$, with the properties that each element of $\mathcal{P}(t_1, t_2, w, w')$ is an interval of waves at time t_2, entropic w.r.t. the flux $\mathtt{f}_k^{\mathrm{eff}}(t_2)$ of Definition 3.37, as follows.

Assume first that $t_2 = j\frac{\varepsilon}{2}, j \in \mathbb{N}$.

(1) If at least one between w, w' does not belong to $\mathcal{W}_k(t_1)$ or $w, w' \in \mathcal{W}_k(t_1)$, but they have never interacted at time t_1, then the equivalence classes of the partition $\mathcal{P}(t_1, t_2, w, w')$ are singletons.

(2) Assume now that w, w' have already interacted at time t_1; we distinguish two cases:
   (a) if $t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, argue by recursion:
   - if $t_2 = t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then $\mathcal{P}(t_1, t_2, , w, w')$ is given by the equivalence relation

$$z \sim z' \quad \Longleftrightarrow \quad \begin{cases} z, z' \text{ are not divided by the Riemann problem} \\ \mathcal{W}_k(t_1, \mathtt{x}(t_1, w)) \text{ with flux function } \mathtt{f}_k^{\mathrm{eff}}(t_1, \cdot); \end{cases}$$

   - if $t_2 = \frac{j\varepsilon}{2} > (j-1)\frac{\varepsilon}{2} \geq t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then $\mathcal{P}(t_1, t_2, w, w')$ is given by the equivalence relation

$$z \sim z' \quad \Longleftrightarrow \quad \begin{cases} \left[\begin{array}{l} z, z' \text{ belong to the same} \\ \text{equivalence class } \mathcal{J} \in \mathcal{P}(t_1, (j-1)\frac{\varepsilon}{2}, w, w') \\ \text{and the Riemann problem } \mathcal{J} \cap \mathcal{W}_k(t_2) \\ \text{with flux } \mathtt{f}_k^{\mathrm{eff}}(t_2) \text{ does not divide them} \end{array} \right] \\ \text{or} \\ \left[\mathtt{t}^{\mathrm{cr}}(z) = \mathtt{t}^{\mathrm{cr}}(z') = t_2 \text{ and } z = z' \right]. \end{cases}$$

   (b) if $t_1 > \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, set

$$\mathcal{P}(t_1, t_2, w, w') = \mathcal{P}(\mathtt{t}^{\mathrm{split}}(t_1, w, w'), t_2, w, w')$$

Finally extend the definition of $\mathcal{P}(t_1,t_2,w,w')$ for any time $t_2 \in \big[\frac{j\varepsilon}{2},(j+1)\frac{\varepsilon}{2}\big)$, $j \in \mathbb{N}$, setting

$$\mathcal{P}(t_1,t_2,w,w') = \mathcal{P}(t_1,i\varepsilon,w,w') \qquad \text{for any } \bar{t} \in \left[\frac{j\varepsilon}{2},(j+1)\frac{\varepsilon}{2}\right).$$

Observe that the previous definition is well posed, provided that $\mathcal{J} \cap \mathcal{W}(i\varepsilon)$ is an interval of waves at time $i\varepsilon$. This will be an easy consequence of Proposition 3.52 and Corollary 3.26, Point (1). Observe also that, while the intervals $\mathcal{I}(t_1,t_2,w,w')$ are constant for $t_2 \in [i\varepsilon,(i+1)\varepsilon)$, the partitions $\mathcal{P}(t_1,t_2,w,w')$ are constant on the intervals $[j\frac{\varepsilon}{2},(j+1)\frac{\varepsilon}{e})$, but they can change at times $j\frac{\varepsilon}{2}$, $j \in \mathbb{N}$.

Remark 3.49. As a consequence of Remark 3.36 we immediately see that each element of the partition $\mathcal{P}(t_1,t_2,w,w')$ is an entropic interval of waves w.r.t. the flux function $\mathtt{f}_k^{\mathrm{eff}}(t_2,\cdot)$ and the relation induced on $\mathcal{P}(t_1,t_2,w,w')$ by the order $\leq$ is still a total order on $\mathcal{P}(t_1,t_2,w,w')$.

Let us prove now some properties of the partition $\mathcal{P}(t_1,t_2,w,w')$.

Lemma 3.50. *Let $w,w',z,z' \in \mathcal{W}_k(t_2)$ be two k-waves. Assume that they satisfy $\mathtt{p}(t_1,w,w')$ and $\mathtt{p}(t_1,z,z')$. If $z \in \mathcal{E}(t_2,w)$, $z' \in \mathcal{E}(t_2,w')$, then*

$$\mathcal{I}(t_1,t_2,w,w') = \mathcal{I}(t_1,t_2,z,z') \qquad \text{and} \qquad \mathcal{P}(t_1,t_2,w,w') = \mathcal{P}(t_1,t_2,z,z').$$

Proof. The proof is an easy consequence of the previous definitions. □

Lemma 3.51. *Let $t_1,t_2,t_2' \in [0,T]$, $0 \leq t_1 \leq t_2 \leq t_2'$. Let $w,w' \in \mathcal{W}_k(t_2) \cap \mathcal{W}_k(t_2')$ and assume that they satisfy $\mathtt{p}(t_1,w,w')$. Let $\mathcal{J} \in \mathcal{P}(t_1,t_2',w,w')$. Then either $\mathcal{J} \cap \mathcal{W}_k(t_2) = \emptyset$ or $\mathcal{J} \cap \mathcal{W}_k(t_2) = \mathcal{J}$ and $\mathcal{J}$ is an interval of waves at time t_2.*

Proof. If at least one between w,w' does not belong to $\mathcal{W}_k(t_1)$ or $w,w' \in \mathcal{W}_k(t_1)$, but they have never interacted at time t_1, then the proof is trivial.

We can thus assume that w,w' have already interacted at time t_1 and, w.l.o.g., we can also assume that $t_1 = \mathtt{t}^{\mathrm{split}}(t_1,w,w')$. It is sufficient to prove the lemma for $t_2,t_2' \in \mathbb{N}\frac{\varepsilon}{2}$. We prove the lemma by induction on times $t_2' \in \mathbb{N}\frac{\varepsilon}{2}$, $t_2' = t_2, t_2 + \frac{\varepsilon}{2}, \dots$.

If $t_2' = t_2$ the proof is trivial. Hence assume that the lemma is proved for time $t_2' - \frac{\varepsilon}{2}$ and let us prove it for time $t_2 \in \mathbb{N}\frac{\varepsilon}{2}$, with $t_1 \leq t_2 < t_2 + \frac{\varepsilon}{2} \leq t_2$. Let $\mathcal{J} \in \mathcal{P}(t_1,t_2',w,w')$ and assume that $\mathcal{J} \cap \mathcal{W}_k(t_2) \neq \emptyset$. Let $z \in \mathcal{J} \cap \mathcal{W}_k(t_2)$, $z' \in \mathcal{J}$. Since $z \sim z'$ at time t_2' and $z \in \mathcal{W}_k(t_2)$, with $t_2 < t_2'$, by definition of equivalence classes, there must be $\mathcal{K} \in \mathcal{P}(t_1,t_2' - \frac{\varepsilon}{2},w,w')$ such that $z,z' \in \mathcal{K}$ and $\mathcal{K} \supseteq \mathcal{J}$. By inductive assumption, $\mathcal{K} \cap \mathcal{W}_k(t_2) = \mathcal{K}$ and thus

$$\mathcal{J} \cap \mathcal{W}_k(t_2) = \mathcal{J} \cap \mathcal{K} \cap \mathcal{W}_k(t_2) = \mathcal{J} \cap \mathcal{K} = \mathcal{J},$$

thus proving the first part of the statement.

Let now $z,z' \in \mathcal{J} \subseteq \mathcal{K}$, $y \in \mathcal{W}_k(t_2)$, $z \leq y \leq z'$. By inductive assumption $y \in \mathcal{K}$; since $\mathcal{K} \cap \mathcal{W}_k(t_2')$ with flux function $\mathtt{f}_k^{\mathrm{eff}}(t_2')$ does not divide z,z' and $z \leq y \leq z'$, we have that $\mathcal{K} \cap \mathcal{W}_k(t_2')$ does not divide z,z',y and thus $y \in \mathcal{J}$, thus proving also the second part of the lemma. □

Proposition 3.52. *Let $t_1,t_2 \in [0,T]$, $t_1 \leq t_2$. Let $w,w' \in \mathcal{W}_k(t_2)$ be two k-waves. Assume that they satisfy $\mathtt{p}(t_1,w,w')$. Let $\mathcal{J} \in \mathcal{P}(t_1,t_2,w,w')$. Then $\mathtt{x}(t_2,\cdot)$ is constant on $\mathcal{J}$.*

Proof. If at least one between w,w' does not belong to $\mathcal{W}_k(t_1)$ or $w,w' \in \mathcal{W}_k(t_1)$, but they have never interacted at time t_1, then the proof is trivial.

We can thus assume that w,w' have already interacted at time t_1 and, w.l.o.g., we can also assume that $t_1 = \mathtt{t}^{\mathrm{split}}(t_1,w,w')$. Clearly it is sufficient to prove the proposition for times $t_2 = \frac{j\varepsilon}{2}, j \in \mathbb{N}$, since, if the proposition is proved at time $\frac{j\varepsilon}{2}$, then it holds also for times $t \in \left[\frac{j\varepsilon}{2}, (j+1)\frac{\varepsilon}{2}\right)$. Hence let $t_2 = \frac{j\varepsilon}{2}$ for some $j \in \mathbb{N}$. Let $\mathcal{J} \in \mathcal{P}(t_1, t_2, w, w')$ and let $z, z' \in \mathcal{J}$. We want to prove that

$$\mathtt{x}(t_2, z) = \mathtt{x}(t_2, z'). \tag{3.42}$$

We argue by induction on i.

(1) If $t_2 = \frac{j\varepsilon}{2} = t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then (3.42) is an immediate consequence of the definition of $\mathcal{P}(t_1, t_2, w, w')$.
(2) If $t_2 = \frac{j\varepsilon}{2} > (j-1)\frac{\varepsilon}{2} \geq t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, two cases arise:
   (a) $\mathtt{t}^{\mathrm{cr}}(z) = \mathtt{t}^{\mathrm{cr}}(z') = t_2$ and $z = z'$; in this case the conclusion is trivial;
   (b) there is $\mathcal{K} \in \mathcal{P}(t_1, (j-1)\frac{\varepsilon}{2}, w, w')$ such that $z, z' \in \mathcal{K}$ and the Riemann problem $\mathcal{K} \cap \mathcal{W}_k(t_2)$ with flux function $\mathtt{f}_k^{\mathrm{eff}}(t_2)$ does not divide z, z' (Point (2a) above); distinguish two more situations:
      (i) j is even: in this case, the conclusion is an immediate consequence of the inductive assumption;
      (ii) j is odd: by inductive assumption, all the waves in $\mathcal{K}$ have the same position (say $m\varepsilon$) at time $\big(j-1\big)\frac{\varepsilon}{2}$ and, by Remark 3.49, $\mathcal{K}$ is entropic w.r.t. the flux $\mathtt{f}_k^{\mathrm{eff}}\big((j-1)\frac{\varepsilon}{2}\big)$; hence the Riemann problem $\mathcal{K} = \mathcal{K} \cap \mathcal{W}_k\big((j-1)\frac{\varepsilon}{2}, m\varepsilon\big)$ does not divide z, z' and thus, by Proposition 1.7, also the Riemann problem $\mathcal{W}_k\big((j-1)\frac{\varepsilon}{2}, m\varepsilon\big)$ does not divide z, z', which implies that $\mathtt{x}(t_2, z) = \mathtt{x}(t_2, z')$. □

Definition 3.53. Let A, B two sets, $A \subseteq B$. Let $\mathcal{P}$ be a partition of B. We say that $\mathcal{P}$ *can be restricted to* A if for any $C \in \mathcal{P}$, either $C \subseteq A$ or $C \subseteq B \setminus A$. We also write

$$\mathcal{P}|_A := \big\{C \in \mathcal{P} \mid C \subseteq A\big\}.$$

Clearly $\mathcal{P}$ can be restricted to A if and only if it can be restricted to $B \setminus A$.

Proposition 3.54. *Let $t_1 \leq t_2$, be two times. Let $w, w', z, z' \in \mathcal{W}_k(t_2)$ be two k-waves, $z \leq w < w' \leq z'$. Assume that they have the same sign and that they satisfy both $\mathtt{p}(t_1, w, w')$ and $\mathtt{p}(t_1, z, z')$. Then $\mathcal{P}(t_1, t_2, z, z')$ can be restricted both to $\mathcal{I}(t_1, t_2, z, z') \cap \mathcal{I}(t_1, t_2, w, w')$ and to $\mathcal{I}(t_1, t_2, z, z') \setminus \mathcal{I}(t_1, t_2, w, w')$.*

Proof. As before, it is sufficient to prove the proposition for times $t_2 = j\frac{\varepsilon}{2}, j \in \mathbb{N}$. If either at least one between z, z' does not belong to $\mathcal{W}_k(t_1)$ or $z, z' \in \mathcal{W}_k(t_1)$ but they have never interacted at time t_1, the proof is immediate being the equivalent classes singletons. Hence, assume that $z, z' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1. We can assume w.l.o.g. that $t_1 = \mathtt{t}^{\mathrm{split}}(t_1, z, z')$. Let $\mathcal{J} \in \mathcal{P}(t_1, t_2, z, z')$ such that $\mathcal{J} \cap \mathcal{I}(t_1, t_2, w, w') \neq \emptyset$. We want to prove that $\mathcal{J} \subseteq \mathcal{I}(t_1, t_2, w, w')$.

Assume first that either at least one between w, w' does not belong to $\mathcal{W}_k(t_1)$ or $w, w' \in \mathcal{W}_k(t_1)$ but they have never interacted at time t_1. Suppose w.l.o.g. that $\mathtt{t}^{\mathrm{cr}}(w) \leq \mathtt{t}^{\mathrm{cr}}(w')$, the case $\mathtt{t}^{\mathrm{cr}}(w) > \mathtt{t}^{\mathrm{cr}}(w')$ being analogous. Since w, w' does not exist at time t_1 or they have never interacted at time t_1, while z, z' have already interacted at time t_1, it must hold $\mathtt{t}^{\mathrm{cr}}(w') > t_1 = \mathtt{t}^{\mathrm{split}}(t_1, z, z')$. It holds

$$\emptyset \neq \mathcal{J} \cap \mathcal{I}(t_1, t_2, w, w') = \Big(\mathcal{J} \cap \Big\{y \in \mathcal{W}_k(t_2) \,\Big|\, \mathcal{S}(y) = \mathcal{S}(w) \text{ and } y < \mathcal{E}(t_2, w')\Big\}\Big) \cup \Big(\mathcal{J} \cap \mathcal{E}(t_2, w')\Big).$$

Distinguish two cases:

(1) if $\mathcal{J} \cap \mathcal{E}(t_2, w') \neq \emptyset$, since $\mathtt{t}^{\mathrm{cr}}(w') > t_1 = \mathtt{t}^{\mathrm{split}}(t_1, z, z')$, $\mathcal{J}$ is a singleton by Point (2a), page 65, and thus $\mathcal{J} \subseteq \mathcal{E}(t_2, w') \subseteq \mathcal{I}(t_1, t_2, w, w')$;

(2) otherwise, if $\mathcal{J} \cap \mathcal{E}(t_2, w') = \emptyset$ and $\mathcal{J} \cap \{y \in \mathcal{W}_k(t_2) \mid \mathcal{S}(y) = \mathcal{S}(w) \text{ and } y < \mathcal{E}(t_2, w')\} \neq \emptyset$, since $\mathcal{J}$ is an interval of waves and $\mathcal{E}(t_2, w') \neq \emptyset$, it must hold $\mathcal{J} \subseteq \{y \in \mathcal{W}_k(t_2) \mid y < \mathcal{E}(t_2, w')\} \subseteq \mathcal{I}(t_1, t_2, w, w')$.

Assume now $w, w' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1. Since $t_1 = \mathtt{t}^{\mathrm{split}}(t_1, z, z')$ and $z \leq w \leq w' \leq z'$, by the monotonicity of the position function, it must hold $\mathtt{x}(t_1, z) = \mathtt{x}(t_1, w) = \mathtt{x}(t_1, w') = \mathtt{x}(t_1, z')$. Moreover, since $\mathtt{p}(t_1, w, w')$ holds, w, w' are divided at time t_1 and thus $\mathtt{t}^{\mathrm{split}}(t_1, w, w') = t_1$. Recall that two waves are divided at a time t if they have different position in a right open neighborhood $(t, t+\eta)$ of t. We argue now by induction on j.

(1) If $t_2 = \frac{j\varepsilon}{2} = t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then $\mathcal{I}(t_1, t_2, w, w') = \mathcal{W}_k(t_1, \mathtt{x}(t_1, w))$ and thus $\mathcal{J} \cap \mathcal{W}_k(t_1, \mathtt{x}(t_1, w)) \neq \emptyset$. By Proposition 3.52, it must hold $\mathcal{J} \subseteq \mathcal{W}_k(t_1, \mathtt{x}(t_1, w)) = \mathcal{I}(t_1, t_2, w, w')$.
(2) If $t_2 = \frac{j\varepsilon}{2} > (j-1)\frac{\varepsilon}{2} \geq t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, assume that the proposition is proved for time $(j-1)\frac{\varepsilon}{2}$. Distinguish two more cases:
   (a) at least one wave in $\mathcal{J}$ is created at time $j\frac{\varepsilon}{2}$; in this case, $\mathcal{J}$ is a singleton and thus $\mathcal{J} \subseteq \mathcal{I}(t_1, t_2, w, w')$;
   (b) all the waves in $\mathcal{J}$ already exist at time $(j-1)\frac{\varepsilon}{2}$; in this case by the definition of $\mathcal{P}\big(t_1, \frac{j\varepsilon}{2}, z, z'\big)$, there is $\mathcal{K} \in \mathcal{P}\big(t_1, (j-1)\frac{\varepsilon}{2}, z, z'\big)$ such that $\mathcal{J} \subseteq \mathcal{K}$. Now observe that

$$\begin{aligned}
\emptyset \neq \mathcal{J} &\cap \mathcal{I}\Big(t_1, \frac{j\varepsilon}{2}, w, w'\Big) \\
&= \mathcal{J} \cap \mathcal{I}\Big(t_1, \frac{j\varepsilon}{2}, w, w'\Big) \cap \mathcal{W}_k\Big((j-1)\frac{\varepsilon}{2}\Big)
\end{aligned}$$

(by Lemma 3.48 and the definition of characteristic interval)

$$\begin{aligned}
&= \mathcal{J} \cap \mathcal{I}\Big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\Big) \cap \mathcal{W}_k\Big(\frac{j\varepsilon}{2}\Big) \\
&\subseteq \mathcal{K} \cap \mathcal{I}\Big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\Big) \cap \mathcal{W}_k\Big(\frac{j\varepsilon}{2}\Big) \\
&\subseteq \mathcal{K} \cap \mathcal{I}\Big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\Big).
\end{aligned}$$

Hence, by inductive assumption, $\mathcal{K} \subseteq \mathcal{I}\big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\big)$ and thus we can conclude, noticing that

$$\begin{aligned}
\mathcal{J} &\subseteq \mathcal{K} \cap \mathcal{W}_k\Big(j\frac{\varepsilon}{2}\Big) \\
&\subseteq \mathcal{I}\Big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\Big). \cap \mathcal{W}_k\Big(j\frac{\varepsilon}{2}\Big) \\
&= \mathcal{I}\Big(t_1, \frac{j\varepsilon}{2}, w, w') \cap \mathcal{W}_k\Big((j-1)\frac{\varepsilon}{2}\Big) \\
&\subseteq \mathcal{I}\Big(t_1, \frac{j\varepsilon}{2}, w, w'\Big),
\end{aligned}$$

where we have again used Lemma 3.48 and the definition of characteristic interval. □

PROPOSITION 3.55. *Let $t_1 \leq t_2$, be two times. Let $w, w', z, z' \in \mathcal{W}_k(t_2)$ be two k-waves, $z \leq w < w' \leq z'$. Assume that they have the same sign and that they satisfy both $\mathtt{p}(t_1, w, w')$ and $\mathtt{p}(t_1, z, z')$.*

(1) *If $w, w' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1, if $z, z' \in \mathcal{I}(t_1, t_2, , w, w')$ and if $\mathtt{t}^{\mathrm{cr}}(z), \mathtt{t}^{\mathrm{cr}}(z') \leq \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then $\mathcal{I}(t_1, t_2, , z, z') = \mathcal{I}(t_1, t_2, , w, w')$ and $\mathcal{P}(t_1, t_2, z, z') = \mathcal{P}(t_1, t_2, w, w')$.*

(2) *If $w, w' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1, but at least one wave between z, z' is created after $\mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then $\mathcal{P}(t_1, t_2, z, z')$ is made by singletons.*

(3) *If either $w, w' \in \mathcal{W}_k(t_1)$ and they have never interacted at time t_1, or if at least one between w, w' does not belong to $\mathcal{W}_k(t_1)$, then $\mathcal{P}(t_1, t_2, z, z')$ is made by singletons.*

PROOF. We prove each point separately.

Proof of Point (1). As before it is sufficient to prove the proposition for times $\frac{j\varepsilon}{2}$, $j \in \mathbb{N}$. We can assume w.l.o.g. that $t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$. If $t_2 = j\frac{\varepsilon}{2} = t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then the proof is obvious. Let thus $t_2 = j\frac{\varepsilon}{2} > (j-1)\frac{\varepsilon}{2} \geq t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$ and assume that the proposition holds at time $(j-1)\frac{\varepsilon}{2}$. If j is odd, then, by inductive assumption,

$$\mathcal{I}\Big(t_1, \frac{j\varepsilon}{2}, w, w'\Big) = \mathcal{I}\Big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\Big) = \mathcal{I}\Big(t_1, (j-1)\frac{\varepsilon}{2}, z, z'\Big) = \mathcal{I}\Big(t_1, \frac{j\varepsilon}{2}, z, z'\Big).$$

If $j = 2i$ is even ($i \in \mathbb{N}$), , then, by Point (2a) of Definition 3.46

$$\begin{aligned}
\mathcal{I}\Big(t_1, j\frac{\varepsilon}{2}, w, w'\Big) &= \mathcal{I}\Big(t_1, i\varepsilon, w, w'\Big) \\
&= \Big\{ y \in \mathcal{W}_k(i\varepsilon) \;\Big|\; \mathcal{S}(y) = \mathcal{S}(w) = \mathcal{S}(w') \\
&\qquad\quad \text{and } \exists\, \tilde{y}, \tilde{y}' \in \mathcal{I}(t_1, (i-1)\varepsilon, w, w') \cap \mathcal{W}_k(i\varepsilon) \text{ such that } \tilde{y} \leq y \leq \tilde{y}' \Big\} \\
&= \Big\{ y \in \mathcal{W}_k(i\varepsilon) \;\Big|\; \mathcal{S}(y) = \mathcal{S}(w) = \mathcal{S}(w') \\
&\qquad\quad \text{and } \exists\, \tilde{y}, \tilde{y}' \in \mathcal{I}\Big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\Big) \cap \mathcal{W}_k(i\varepsilon) \text{ such that } \tilde{y} \leq y \leq \tilde{y}' \Big\} \\
\text{(recursion)} &= \Big\{ y \in \mathcal{W}_k(i\varepsilon) \;\Big|\; \mathcal{S}(y) = \mathcal{S}(z) = \mathcal{S}(z') \\
&\qquad\quad \text{and } \exists\, \tilde{y}, \tilde{y}' \in \mathcal{I}\Big(t_1, (j-1)\frac{\varepsilon}{2}, z, z'\Big) \cap \mathcal{W}_k(i\varepsilon) \text{ such that } \tilde{y} \leq y \leq \tilde{y}' \Big\} \\
&= \Big\{ y \in \mathcal{W}_k(i\varepsilon) \;\Big|\; \mathcal{S}(y) = \mathcal{S}(w) = \mathcal{S}(w') \\
&\qquad\quad \text{and } \exists\, \tilde{y}, \tilde{y}' \in \mathcal{I}(t_1, (i-1)\varepsilon, z, z') \cap \mathcal{W}_k(i\varepsilon) \text{ such that } \tilde{y} \leq y \leq \tilde{y}' \Big\} \\
&= \mathcal{I}\Big(t_1, i\varepsilon, z, z'\Big) \\
&= \mathcal{I}\Big(t_1, j\frac{\varepsilon}{2}, z, z'\Big).
\end{aligned}$$

Now assume that $y, y' \in \mathcal{I}\big(t_1, \frac{j\varepsilon}{2}, w, w'\big) = \mathcal{I}\big(t_1, \frac{j\varepsilon}{2}, z, z'\big)$. Then it holds

$$\begin{aligned}
y \sim y' \text{ w.r.t. } \mathcal{P}\Big(t_1, \frac{j\varepsilon}{2}, w, w'\Big) \iff & \left\{\begin{array}{l} \left[\begin{array}{l} y, y' \text{ belong to the same equivalence} \\ \text{class } \mathcal{J} \in \mathcal{P}\Big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\Big) \text{ at time } (j-1)\frac{\varepsilon}{2} \\ \text{and the Riemann problem } \mathcal{J} \cap \mathcal{W}\Big(\frac{j\varepsilon}{2}\Big) \\ \text{with flux } \mathtt{f}_k^{\mathrm{eff}}\Big(\frac{j\varepsilon}{2}\Big) \text{ does not divide them} \end{array}\right] \\ \text{or} \\ \left[\ \mathtt{t}^{\mathrm{cr}}(y) = \mathtt{t}^{\mathrm{cr}}(y') \text{ and } y = y'\ \right] \end{array}\right. \\
& \Big(\text{by } \mathcal{P}\Big(t_1, (j-1)\frac{\varepsilon}{2}, w, w'\Big) = \mathcal{P}\Big(t_1, (j-1)\frac{\varepsilon}{2}, z, z'\Big)\Big) \\
\iff & \left\{\begin{array}{l} \left[\begin{array}{l} y, y' \text{ belong to the same equivalence} \\ \text{class } \mathcal{J} \in \mathcal{P}\Big(t_1, (j-1)\frac{\varepsilon}{2}, z, z'\Big) \text{ at time } (j-1)\frac{\varepsilon}{2} \\ \text{and the Riemann problem } \mathcal{J} \cap \mathcal{W}\Big(\frac{j\varepsilon}{2}\Big) \\ \text{with flux function } \mathtt{f}_k^{\mathrm{eff}}\Big(\frac{j\varepsilon}{2}\Big) \text{ does not divide them} \end{array}\right] \\ \text{or} \\ \left[\ \mathtt{t}^{\mathrm{cr}}(y) = \mathtt{t}^{\mathrm{cr}}(y') \text{ and } y = y'\ \right] \end{array}\right. \\
\iff & \ y \sim y' \text{ w.r.t. the partition } \mathcal{P}\Big(t_1, \frac{j\varepsilon}{2}, z, z'\Big).
\end{aligned}$$

Hence $\mathcal{P}\Big(t_1, \frac{j\varepsilon}{2}, w, w'\Big) = \mathcal{P}\Big(t_1, \frac{j\varepsilon}{2}, z, z'\Big)$.

Proof of Point (2). Let us now prove the second point, assuming w.l.o.g. that $\mathtt{t}^{\mathrm{cr}}(z) > \mathtt{t}^{\mathrm{split}}(t_1, w, w')$. Assume by contradiction that $\mathcal{P}(t_1, t_2, z, z')$ contains at least one element which is not a singleton. Then $z, z' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1. This means that there exists a time $\tilde{t} \leq t_1$ such that $\mathtt{x}(\tilde{t}, z) = \mathtt{x}(\tilde{t}, z')$. Clearly $\tilde{t} \geq \mathtt{t}^{\mathrm{cr}}(z) > \mathtt{t}^{\mathrm{split}}(\bar{t}, w, w')$. Therefore, at time $\tilde{t}$, $w, w', z, z' \in \mathcal{W}_k(\tilde{t})$ and thus, by the monotonicity of $\mathtt{x}$, it should happen $x(\tilde{t}, z) = \mathtt{x}(\tilde{t}, w) = \mathtt{x}(\tilde{t}, w') = \mathtt{x}(\tilde{t}, z')$, a contradiction, since $t_1 \geq \tilde{t} \geq \mathtt{t}^{\mathrm{cr}}(z) > \mathtt{t}^{\mathrm{split}}(\bar{t}, w, w')$.

Proof of Point (3). Let us now prove the third part of the proposition. We consider only the case $\mathtt{t}^{\mathrm{cr}}(w) \leq \mathtt{t}^{\mathrm{cr}}(w')$, the case $\mathtt{t}^{\mathrm{cr}}(w) > \mathtt{t}^{\mathrm{cr}}(w')$ being completely similar. Assume by contradiction that $\mathcal{P}(t_1, t_2, z, z')$ contains at least one element which is not a singleton. Then $z, z' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1. This means that there is a time $\tilde{t} \leq t_1$ such that $\mathtt{x}(\tilde{t}, z) = \mathtt{x}(\tilde{t}, z')$. Since $z' \in \mathcal{E}(t_2, w')$, it must hold $t_2 \geq t_1 \geq \tilde{t} \geq \mathtt{t}^{\mathrm{cr}}(z') = \mathtt{t}^{\mathrm{cr}}(w') \geq \mathtt{t}^{\mathrm{cr}}(w)$. Hence $w, w' \in \mathcal{W}_k(t_1)$, $w, w', z, z' \in \mathcal{W}_k(\tilde{t})$ and by the monotonicity of $\mathtt{x}$, we have $\mathtt{x}(z) = \mathtt{x}(w) = \mathtt{x}(w') = \mathtt{x}(z')$, a contradiction. $\square$

3.5. The quadratic interaction potential

Now we have all the tools we need to define the functional $\mathfrak{Q}_k$ (for every $k = 1, \dots, n$) and to prove that it satisfies the inequality (3.5), thus obtaining the *global* part of the proof of Theorem A.

In Section 3.5.1 we give the definition of $\mathfrak{Q}_k$, using the intervals $\mathcal{I}(t_1, t_2, w, w')$ and their partitions $\mathcal{P}(t_1, t_2, w, w')$. In Section 3.5.2 we state the main theorem of this last part of the paper, i.e. inequality (3.5) and we give a sketch of its proof, which will be written down in details in Sections 3.5.3, 3.5.4, 3.5.5.

3.5.1. Definition of the functional $\mathfrak{Q}$. We define now for each family $k = 1, \dots, n$, the functional $\mathfrak{Q}_k = \mathfrak{Q}_k(t)$, which bounds the change in speed of the waves in the approximate solution u^ε, or more precisely, which satisfies (3.5).

We first define the weight $\mathfrak{q}_k(t, w, w')$ of a pair of waves (w, w') at time t as follows. First of all, fix three times $t_1 \leq t_2 \leq t_3$. Assume that $w, w' \in \mathcal{W}_k(t_2) \cap \mathcal{W}_k(t_3)$ and that $\mathtt{p}(t_1, w, w')$ holds. We define the *weight of (w, w') at time t_2, starting from time t_1 and ending at time t_3* as

$$\mathfrak{q}_k(t_1, t_2, t_3, w, w') := \frac{\pi_k(t_1, t_2, t_3, w, w')}{d_k(t_1, t_2, t_3, w, w')}, \tag{3.43}$$

where $\pi_k(t_1, t_2, t_3, w, w')$, $d_k(t_1, t_2, t_3, w, w')$ are defined as follows. Let

$$\begin{aligned} &\mathcal{J}, \mathcal{J}' \in \mathcal{P}(t_1, t_2, w, w'), \text{ such that } w \in \mathcal{J}, w' \in \mathcal{J}', \\ &\mathcal{K}, \mathcal{K}' \in \mathcal{P}(t_1, t_3, w, w'), \text{ such that } w \in \mathcal{K}, w' \in \mathcal{K}' \end{aligned} \tag{3.44}$$

be the elements of the partition of $\mathcal{I}(t_1, t_2, w, w')$ and $\mathcal{I}(t_1, t_3, w, w')$ containing w, w' respectively. Set

$$\mathcal{G} := \mathcal{K} \cup \big\{ z \in \mathcal{J} \;\big|\; z > \mathcal{K} \big\}, \qquad \mathcal{G}' := \mathcal{K}' \cup \big\{ z \in \mathcal{J}' \;\big|\; z < \mathcal{K}' \big\}, \tag{3.45}$$

and

$$\mathcal{B} := \mathcal{K} \cup \Big\{ z \in \mathcal{W}_k(t_2) \;\Big|\; \mathcal{S}(z) = \mathcal{S}(w) = \mathcal{S}(w') \text{ and } \mathcal{K} < z < \mathcal{K}' \Big\} \cup \mathcal{K}'.$$

By Lemma 3.51 $\mathcal{G}, \mathcal{G}'$ are i.o.w.s at time t_2. We can thus define

$$\pi_k(t_1, t_2, t_3, w, w') := \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{G}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{G}') \Big]^+ \tag{3.46}$$

and

$$d_k(t_1, t_2, t_3, w, w') := \mathcal{L}^1\big(\mathcal{B}\big). \tag{3.47}$$

REMARK 3.56. It is easy to see that $\mathfrak{q}_k(t_1, t_2, t_3, w, w')$ is uniformly bounded: in fact,

$$0 \leq \mathfrak{q}_k(t_1, t_2, t_3, w, w') = \frac{\pi_k(t_1, t_2, t_3, w, w')}{d_k(t_1, t_2, t_3, w, w')} \leq \|D^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)\|_\infty \leq \mathcal{O}(1).$$

Moreover, by the definition of the characteristic intervals and their partitions, if $w, w' \in \mathcal{W}_k(t_1)$ are divided and have already interacted, then

$$\mathfrak{q}_k(t_1, t_2, t_3, w, w') = \mathfrak{q}_k\big(\mathtt{t}^{\mathrm{split}}(t_1, w, w'), t_2, t_3, w, w'\big).$$

Fix now two times $t_1 \leq t_2$ such that $w, w' \in \mathcal{W}_k(t_2)$ and $\mathtt{p}(t_1, w, w')$ holds. Define the *weight of (w, w') at time t_2 starting from time t_1* as

$$\mathfrak{q}_k(t_1, t_2, w, w') := \sup_{\substack{t_3 \geq t_2 \\ w, w' \in \mathcal{W}_k(t_3)}} \mathfrak{q}_k(t_1, t_2, t_3, w, w'). \tag{3.48}$$

Observe that the above sup is actually a max, since it is the supremum of a finite set (we are working in the time interval $[0, T]$).

Finally, for any fixed time t and for any $w, w' \in \mathcal{W}_k(t)$, define the *weight of (w, w') at time t* as

$$\mathfrak{q}_k(t, w, w') := \begin{cases} \mathfrak{q}_k(t, t, w, w'), & \text{if } w, w' \text{ are divided in the real solution at time } t, \\ 0, & \text{otherwise.} \end{cases} \tag{3.49}$$

We can finally define the functional $\mathfrak{Q}_k(t)$ as

$$\mathfrak{Q}_k(t) := \mathfrak{Q}_k^+(t) + \mathfrak{Q}_k^-(t),$$

where

$$\mathfrak{Q}_k^+(t) := \int_{\{(w,w') \in \mathcal{W}_k^+(t) \times \mathcal{W}_k^+(t) \ | \ w<w'\}} \mathtt{q}_k(t,w,w')dwdw'$$

and

$$\mathfrak{Q}_k^-(t) := \int_{\{(w,w') \in \mathcal{W}_k^-(t) \times \mathcal{W}_k^-(t) \ | \ w<w'\}} \mathtt{q}_k(t,w,w')dwdw'$$

REMARK 3.57. Clearly $\mathfrak{Q}_k(t)$ is constant on the time intervals $\big[\frac{j\varepsilon}{2}, \frac{(j+1)\varepsilon}{2}\big)$ and it changes its value only at times $\frac{j\varepsilon}{2}$, $j \in \mathbb{N}$.

3.5.2. Statement of the main theorem and sketch of the proof. We now state the main theorem of this last part of the paper and give a sketch of its proof: with this theorem, the proof of the Theorem A is completed.

THEOREM 3.58. *For any $i \in \mathbb{N}$, $i \geq 1$, it holds*

$$\mathfrak{Q}_k(i\varepsilon) - \mathfrak{Q}_k((i-1)\varepsilon) \leq - \sum_{m \in \mathbb{Z}} \mathtt{A}_k^{\text{quadr}}(i\varepsilon, m\varepsilon) + \mathcal{O}(1)\text{Tot.Var.}(u(0);\mathbb{R}) \sum_{m \in \mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon). \tag{3.50}$$

As an immediate consequence of the previous theorem and Theorem 2.15, we get the following corollary.

COROLLARY 3.59. *There exists a constant $M = M(f) > 0$, depending only on f such that the functional*

$$t \mapsto \Upsilon(t) := \mathfrak{Q}(t) + MQ^{\text{known}}(t)$$

is uniformly bounded at $t = 0$:

$$\Upsilon(0) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u}),$$

it is decreasing and at each time step $i\varepsilon$, $i \in \mathbb{N}$, it decreases at least of

$$\frac{1}{2} \sum_{m \in \mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \leq \Upsilon((i-1)\varepsilon) - \Upsilon(i\varepsilon). \tag{3.51}$$

As a consequence,

$$\sum_{m \in \mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u}). \tag{3.52}$$

SKETCH OF THE PROOF OF THEOREM 3.58. First of all observe that it is sufficient to prove inequality (3.50) separately for $\mathfrak{Q}_k^+$ and $\mathfrak{Q}_k^-$. In particular, we will prove only that

$$\mathfrak{Q}_k^+(i\varepsilon) - \mathfrak{Q}_k^+((i-1)\varepsilon) \leq - \sum_{\substack{m \in \mathbb{Z} \\ \mathcal{S}(\mathcal{W}_k(i\varepsilon, m\varepsilon))=1}} \mathtt{A}_k^{\text{quadr}}(i\varepsilon, m\varepsilon) + \mathcal{O}(1)\text{Tot.Var.}(u(0);\mathbb{R}) \sum_{m \in \mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon),$$

since the proof of the same inequality for $\mathfrak{Q}_k^-$ is completely similar.

For any $m \in \mathbb{Z}$, set

$$\begin{aligned}
\mathcal{J}_m^L &:= \mathcal{W}_k^{(1)}\big((i-1)\varepsilon, (m-1)\varepsilon\big) \cap \mathcal{W}_k^+((i-1)\varepsilon), \\
\mathcal{J}_m^R &:= \mathcal{W}_k^{(0)}\big((i-1)\varepsilon, m\varepsilon\big) \cap \mathcal{W}_k^+((i-1)\varepsilon), \\
\mathcal{J}_m &:= \mathcal{J}_m^L \cup \mathcal{J}_m^R, \\
\mathcal{K}_m &:= \mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k^+(i\varepsilon), \\
\mathcal{T}_m &:= \mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k^+((i-1)\varepsilon).
\end{aligned} \tag{3.53}$$

The sets $\mathcal{J}_m^L$ and $\mathcal{J}_m^R$ are the sets of positive waves interacting at point $(i\varepsilon, m\varepsilon)$, coming from the left and from the right respectively. The set $\mathcal{K}_m$ is the set of positive waves located at $\mathcal{W}_k(i\varepsilon, m\varepsilon)$ (thus either it coincide with $\mathcal{W}_k(i\varepsilon, m\varepsilon)$ or it is empty) and finally $\mathcal{T}_m$ is the set of all positive waves in $\mathcal{W}_k(i\varepsilon, m\varepsilon)$ which already exists at time $(i-1)\varepsilon$.

Observe that if $w, w' \in \mathcal{J}_m^L$ (or $w, w' \in \mathcal{J}_m^R$), then w, w' are not divided in the real solution at time $(i-1)\varepsilon$ and thus $\mathfrak{q}_k((i-1)\varepsilon, w, w') = 0$.
Similarly, if $w, w' \in \mathcal{K}_m$, $w < w'$, then either w, w' are not divided at time $i\varepsilon$, and thus $\mathfrak{q}_k(i\varepsilon, w, w') = 0$, or they are divided at time $i\varepsilon$, i.e. they have different positions at times $t \in (i\varepsilon, (i+1)\varepsilon)$. In this second case, by definition $\mathtt{t}^{\text{split}}(i\varepsilon, w, w') = i\varepsilon$; for any fixed time $t_3 \geq i\varepsilon$, with $w, w' \in \mathcal{W}_k(t_3)$, with notations similar to (3.44)-(3.45), denote by

$$\begin{aligned}
&\mathcal{J}, \mathcal{J}' \in \mathcal{P}(i\varepsilon, i\varepsilon, w, w'), \text{ such that } w \in \mathcal{J}, w' \in \mathcal{J}', \\
&\mathcal{K}, \mathcal{K}' \in \mathcal{P}(i\varepsilon, t_3, w, w'), \text{ such that } w \in \mathcal{K}, w' \in \mathcal{K}'.
\end{aligned}$$

the element of the partition containing w, w' at time $i\varepsilon$ and at time t_3 respectively, and set

$$\mathcal{G} := \mathcal{K} \cup \big\{z \in \mathcal{J} \bigm| z > \mathcal{K}\big\}, \qquad \mathcal{G}' := \mathcal{K}' \cup \big\{z \in \mathcal{J}' \bigm| z < \mathcal{K}'\big\}.$$

Using the monotonicity properties of the derivative of the convex envelope and the fact that the element of the partition $\mathcal{P}(i\varepsilon, i\varepsilon, w, w')$ are entropic w.r.t. the function $\mathtt{f}_k^{\text{eff}}(i\varepsilon)$, we obtain

$$0 \geq \sigma^{\text{rh}}(\mathtt{f}_k^{\text{eff}}(i\varepsilon), \mathcal{J}) - \sigma^{\text{rh}}(\mathtt{f}_k^{\text{eff}}(i\varepsilon), \mathcal{J}') \geq \sigma^{\text{rh}}(\mathtt{f}_k^{\text{eff}}(i\varepsilon), \mathcal{G}) - \sigma^{\text{rh}}(\mathtt{f}_k^{\text{eff}}(i\varepsilon), \mathcal{G}').$$

Thus $\pi_k(i\varepsilon, i\varepsilon, t_3, w, w') = 0 = \mathfrak{q}_k(i\varepsilon, i\varepsilon, t_3, w, w')$, for any $t_3 \geq i\varepsilon$ such that $w, w' \in \mathcal{W}_k(t_3)$. Hence, by (3.48) and (3.49),

$$\mathfrak{q}_k(i\varepsilon, w, w') = \mathfrak{q}_k(i\varepsilon, i\varepsilon, w, w') = \sup_{\substack{t_3 \geq i\varepsilon \\ w, w' \in \mathcal{W}_k(j\varepsilon)}} \mathfrak{q}_k(i\varepsilon, i\varepsilon, t_3, w, w') = 0.$$

We can thus perform the following computation:

$$\begin{aligned}
\mathfrak{Q}_k^+(i\varepsilon) - \mathfrak{Q}_k^+((i-1)\varepsilon) \leq{}& \sum_{m<m'} \bigg\{ \iint_{(\mathcal{K}_m \times \mathcal{K}_{m'}) \setminus (\mathcal{T}_m \times \mathcal{T}_{m'})} \mathfrak{q}_k(i\varepsilon) dw dw' \\
&\qquad\qquad + \iint_{\mathcal{T}_m \times \mathcal{T}_{m'}} \Big[\mathfrak{q}_k(i\varepsilon) - \mathfrak{q}_k\big((i-1)\varepsilon\big)\Big] dw dw' \bigg\} \\
&+ \sum_{m \in \mathbb{Z}} \iint_{\mathcal{J}_m^L \times \mathcal{J}_m^R} \bigg[\mathfrak{q}_k\Big(\Big(i - \frac{1}{2}\Big)\varepsilon\Big) - \mathfrak{q}_k\Big((i-1)\varepsilon\Big)\bigg] dw dw' \\
&- \sum_{m \in \mathbb{Z}} \iint_{\mathcal{J}_m^L \times \mathcal{J}_m^R} \mathfrak{q}_k\Big(\Big(i - \frac{1}{2}\Big)\varepsilon\Big) dw dw.
\end{aligned}$$

We will now separately study:

(1) in Section 3.5.3, the integral over *pairs of waves such that at least one of them is created at time* $i\varepsilon$:

$$\iint_{(\mathcal{K}_m \times \mathcal{K}_{m'}) \setminus (\mathcal{T}_m \times \mathcal{T}_{m'})} \mathfrak{q}_k(i\varepsilon) dw dw' \leq \mathcal{O}(1) \text{Tot.Var.}(u(0)) \sum_{m \in \mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon). \tag{3.54}$$

(2) in Section 3.5.4, the variation of the integral over *pairs of waves which exist both at time* $(i-1)\varepsilon$ *and at time* $i\varepsilon$:

$$\sum_{m<m'} \bigg\{ \iint_{\mathcal{T}_m \times \mathcal{T}_{m'}} \Big[\mathfrak{q}_k(i\varepsilon) - \mathfrak{q}_k\big((i-1)\varepsilon\big)\Big] dw dw' \leq \mathcal{O}(1) \text{Tot.Var.}(u(0)) \sum_{r \in \mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon). \tag{3.55}$$

and the variation of the integral *between time* $(i-1)\varepsilon$ *and time* $\left(i-\frac{1}{2}\right)\varepsilon$ *of the pairs of waves interacting at time* $i\varepsilon$:

$$\sum_{m\in\mathbb{Z}} \iint_{\mathcal{J}_m^L\times\mathcal{J}_m^R} \left[\mathfrak{q}_k\Big(\Big(i-\frac{1}{2}\Big)\varepsilon\Big) - \mathfrak{q}_k\Big(\big(i-1\big)\varepsilon\Big)\right] dw dw' \leq \mathcal{O}(1)\text{Tot.Var.}(u(0)) \sum_{r\in\mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon). \tag{3.56}$$

(3) in Section 3.5.5, the (negative) term, related to *pairs of waves which are divided at time* $(i-1)\varepsilon$ *and are interacting at time* $i\varepsilon$:

$$\begin{aligned}
-\sum_{m\in\mathbb{Z}} \iint_{\mathcal{J}_m^L\times\mathcal{J}_m^R} &\mathfrak{q}_k\Big(\Big(i-\frac{1}{2}\Big)\varepsilon\Big) dw dw \\
&\leq -\sum_{\substack{m\in\mathbb{Z} \\ \mathcal{S}(\mathcal{W}_k(i\varepsilon,m\varepsilon))=1}} \mathtt{A}_k^{\text{quadr}}(i\varepsilon, m\varepsilon) + \mathcal{O}(1)\text{Tot.Var.}(u(0)) \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon).
\end{aligned} \tag{3.57}$$

It is easy to see that inequality (3.50) in the statement of Theorem 3.58 follows from (3.54), (3.55), (3.56), (3.57). □

3.5.3. Analysis of pairs with at least one created wave. The integral over pair of waves such that at least one of them is created at time $i\varepsilon$ is estimated in the following proposition.

PROPOSITION 3.60. *It holds*

$$\sum_{m<m'} \iint_{(\mathcal{K}_m\times\mathcal{K}_{m'})\setminus(\mathcal{T}_m\times\mathcal{T}_{m'})} \mathfrak{q}_k(i\varepsilon, w, w') dw dw' \leq \mathcal{O}(1)\text{Tot.Var.}(u(0)) \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon).$$

PROOF. In fact,

$$\begin{aligned}
\mathcal{L}^2\big((\mathcal{K}_m\times\mathcal{K}_{m'})\setminus(\mathcal{T}_m\times\mathcal{T}_{m'})\big) &\leq \mathcal{L}^2\big((\mathcal{K}_m\setminus\mathcal{T}_m)\times\mathcal{K}_{m'}\big) + \mathcal{L}^2\big(\mathcal{K}_m\times(\mathcal{K}_{m'}\setminus\mathcal{T}_{m'})\big) \\
&\leq \mathcal{L}^1(\mathcal{K}_{m'})\mathcal{L}^1(\mathcal{K}_m\setminus\mathcal{T}_m) + \mathcal{L}^1(\mathcal{K}_m)\mathcal{L}^1(\mathcal{K}_{m'}\setminus\mathcal{T}_{m'}).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{m<m'} \iint_{(\mathcal{K}_m\times\mathcal{K}_{m'})\setminus(\mathcal{T}_m\times\mathcal{T}_{m'})} &\mathfrak{q}_k(i\varepsilon, \tau, \tau') d\tau d\tau' \\
&\leq \mathcal{O}(1) \sum_{m<m'} \mathcal{L}^2\big((\mathcal{K}_m\times\mathcal{K}_{m'})\setminus(\mathcal{T}_m\times\mathcal{T}_{m'})\big) \\
&\leq \mathcal{O}(1) \sum_{m<m'} \mathcal{L}^1(\mathcal{K}_{m'})\mathcal{L}^1(\mathcal{K}_m\setminus\mathcal{T}_m) + \mathcal{L}^1(\mathcal{K}_m)\mathcal{L}^1(\mathcal{K}_{m'}\setminus\mathcal{T}_{m'}) \\
&\leq \mathcal{O}(1) \sum_{m'\in\mathbb{Z}} \mathcal{L}^1(\mathcal{K}_{m'}) \sum_{m\in\mathbb{Z}} \mathcal{L}^1(\mathcal{K}_m\setminus\mathcal{T}_m) \\
&\leq \mathcal{O}(1) V_k^+(i\varepsilon) \sum_{m\in\mathbb{Z}} \mathtt{A}_k^{\text{cr}}(i\varepsilon, m\varepsilon) \\
\text{(by (2.18), (2.19) and Corollary 2.10)} &\leq \mathcal{O}(1)\text{Tot.Var.}(u(0)) \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon).
\end{aligned}$$

□

3.5.4. Analysis of pairs of waves which exist both at time $(i-1)\varepsilon$ and at time $i\varepsilon$ and pairs of interacting waves after transversal interactions. The aim of this section is to estimate the variation of the integral over pair of waves which exist both at time $(i-1)\varepsilon$ and at time $i\varepsilon$ and the variation of the integral over pairs of interacting waves between time $(i-1)\varepsilon$ and time $\big(i-\frac{1}{2}\big)\varepsilon$. More precisely we prove the following theorems.

THEOREM 3.61. *For the integral over pairs of waves which exist both at time $(i-1)\varepsilon$ and at time $i\varepsilon$ and which do not interact at time $i\varepsilon$, it holds*

$$\sum_{m<m'}\left\{\iint_{\mathcal{T}_m\times\mathcal{T}_{m'}}\Big[\mathfrak{q}_k(i\varepsilon)-\mathfrak{q}_k\big((i-1)\varepsilon\big)\Big]dwdw'\le\mathcal{O}(1)\text{Tot.Var.}(u(0))\sum_{r\in\mathbb{Z}}\mathtt{A}(i\varepsilon,r\varepsilon).\right. \tag{3.58}$$

THEOREM 3.62. *For pair of waves which are interacting at time $i\varepsilon$, the variation of the integral between time $(i-1)\varepsilon$ and time $\big(i-\frac{1}{2}\big)\varepsilon$ is estimated by the following inequality:*

$$\sum_{m\in\mathbb{Z}}\iint_{\mathcal{J}^L_m\times\mathcal{J}^R_m}\left[\mathfrak{q}_k\Big(\Big(i-\frac{1}{2}\Big)\varepsilon\Big)-\mathfrak{q}_k\Big((i-1)\varepsilon\Big)\right]dwdw'\le\mathcal{O}(1)\text{Tot.Var.}(u(0))\sum_{r\in\mathbb{Z}}\mathtt{A}(i\varepsilon,r\varepsilon).$$

We first need a preliminary result, namely the following lemma, which estimates the change of the numerator π_k and the denominator d_k in the definition of $\mathfrak{q}_k$, formulas (3.46) and (3.47).

LEMMA 3.63. *Let $t_1\le t_2-\frac{\varepsilon}{2}\le t_2\le t_3$. Let $w,w'\in\mathcal{W}_k\big(t_2-\frac{\varepsilon}{2}\big)\cap\mathcal{W}_k(t_2)\cap\mathcal{W}_k(t_3)$, $w<w'$. Assume that $\mathtt{p}(t_1,w,w')$ holds. Let $i:=\min\{j\in\mathbb{N}\mid j\varepsilon\ge t_2\}$. Set*

$$m\varepsilon:=\mathtt{x}(i\varepsilon,w),\qquad m'\varepsilon:=\mathtt{x}(i\varepsilon,w').$$

Setting, for simplicity,

$$\begin{aligned}
\Delta d_k(w,w')&:=d_k\big(t_1,t_2,t_3,w,w'\big)-d_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big),\\
\Delta\pi_k(w,w')&:=p_k\big(t_1,t_2,t_3,w,w'\big)-p_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big),\\
\Delta\mathfrak{q}_k(w,w')&:=\mathfrak{q}_k\big(t_1,t_2,t_3,w,w'\big)-\mathfrak{q}_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big),
\end{aligned}$$

the following inequalities hold:

$$\big|\Delta d_k(w,w')\big|\le\mathcal{O}(1)\sum_{r=m}^{m'}\mathtt{A}(i\varepsilon,r\varepsilon), \tag{3.59a}$$

$$\Delta\pi_k(w,w')\le\mathcal{O}(1)\sum_{r=m}^{m'}\mathtt{A}(i\varepsilon,r\varepsilon), \tag{3.59b}$$

$$\Delta\mathfrak{q}_k(w,w')\le\mathcal{O}(1)\frac{\sum_{r=m}^{m'}\mathtt{A}(i\varepsilon,r\varepsilon)}{\Phi_k\big(t_2-\frac{\varepsilon}{2}\big)(w')-\Phi_k\big(t_2-\frac{\varepsilon}{2}\big)(w)}. \tag{3.59c}$$

PROOF. Let

$$\begin{aligned}
&\mathcal{J},\mathcal{J}'\in\mathcal{P}\Big(t_1,t_2-\frac{\varepsilon}{2},w,w'\Big), && w\in\mathcal{J},w'\in\mathcal{J}',\\
&\tilde{\mathcal{J}},\tilde{\mathcal{J}}'\in\mathcal{P}(t_1,t_2,w,w'), && w\in\tilde{\mathcal{J}},w'\in\tilde{\mathcal{J}}',\\
&\mathcal{K},\mathcal{K}'\in\mathcal{P}(t_1,t_3,w,w'), && w\in\mathcal{K},w'\in\mathcal{K}'.
\end{aligned}$$

Set also

$$\mathcal{A}:=\mathcal{K}\cup\left\{z\in\mathcal{W}_k^+\Big(t_2-\frac{\varepsilon}{2}\Big)\;\middle|\;\mathcal{K}<z<\mathcal{K}'\right\}\cup\mathcal{K}',\quad\mathcal{B}:=\mathcal{K}\cup\big\{z\in\mathcal{W}_k^+(t_2)\;\big|\;\mathcal{K}<z<\mathcal{K}'\big\}\cup\mathcal{K}'.$$

It is easy to see that

$$\mathcal{A} \subseteq \bigcup_{r=m}^{m'} \Big\{ w \in \mathcal{W}_k\Big(t_2 - \frac{\varepsilon}{2}\Big) \;\Big|\; \mathtt{x}(i\varepsilon, w) = r\varepsilon \Big\}, \quad \mathcal{B} \subseteq \bigcup_{r=m}^{m'} \Big\{ w \in \mathcal{W}_k(t_2) \;\Big|\; \mathtt{x}(i\varepsilon, w) = r\varepsilon \Big\}, \tag{3.60}$$

Observe also that

$$\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap \mathcal{W}_k(t_2) = \mathcal{W}_k\Big(t_2 - \frac{\varepsilon}{2}\Big) \cap \mathcal{B}. \tag{3.61}$$

Moreover, the bijection

$$\Theta : \Phi_k\Big(t_2 - \frac{\varepsilon}{2}\Big)\Big(\mathcal{A} \cap \mathcal{B}\Big) \to \Phi_k(t_2)\Big(\mathcal{A} \cap \mathcal{B}\Big)$$

defined as

$$\Theta := \Phi_k(t_2) \circ \Phi_k\Big(t_2 - \frac{\varepsilon}{2}\Big)$$

satisfies, by Proposition 3.31,

$$\Theta_\sharp\Big(\mathcal{L}^1|_{\Phi_k(t_2-\varepsilon/2)(\mathcal{A}\cap\mathcal{B})}\Big) = \mathcal{L}^1|_{\Phi_k(t_2)(\mathcal{A}\cap\mathcal{B})}. \tag{3.62}$$

We will use the map Θ to compare the effective fluxes at times $t_2 - \frac{\varepsilon}{2}$ and t_2.

We now prove separately the two inequalities of the statement.

Proof of (3.59a). We have

$$\begin{aligned}
\big|\Delta d_k(w, w')\big| &= \Big| d(t_1, t_2, t_3, w, w') - d\Big(t_1, t_2 - \frac{\varepsilon}{2}, t_3, w, w'\Big)\Big| \\
&= \Big| \mathcal{L}^1(\mathcal{B}) - \mathcal{L}^1(\mathcal{A}) \Big| \\
&= \mathcal{L}^1(\mathcal{B} \setminus \mathcal{A}) + \mathcal{L}^1(\mathcal{A} \setminus \mathcal{B}) \\
\text{(by (3.61))} \quad &= \mathcal{L}^1\Big(\mathcal{B} \setminus \mathcal{W}_k\Big(t_2 - \frac{\varepsilon}{2}\Big)\Big) + \mathcal{L}^1(\mathcal{A} \setminus \mathcal{W}_k(t_2)) \\
\text{(by (3.60))} \quad &\le \sum_{r=m}^{m'} \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) + \mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, r\varepsilon) \\
\text{(by Cor. 2.10)} \quad &\le \sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon).
\end{aligned}$$

Proof of (3.59b). The proof of (3.59b) is more involved. Define

$$\mathcal{F} := \mathcal{K} \cup \Big(\mathcal{J} \cap \Big\{z \in \mathcal{W}_k\Big(t_2 - \frac{\varepsilon}{2}\Big) \mid z > \mathcal{K}\Big\}\Big), \quad \mathcal{F}' := \mathcal{K}' \cup \Big(\mathcal{J}' \cap \Big\{z \in \mathcal{W}_k\Big(t_2 - \frac{\varepsilon}{2}\Big) \mid z < \mathcal{K}'\Big\}\Big),$$

$$\mathcal{G} := \mathcal{K} \cup \Big(\tilde{\mathcal{J}} \cap \big\{z \in \mathcal{W}_k(t_2) \mid z > \mathcal{K}\big\}\Big), \qquad \mathcal{G}' := \mathcal{K}' \cup \Big(\tilde{\mathcal{J}}' \cap \big\{z \in \mathcal{W}_k(t_2) \mid z < \mathcal{K}'\big\}\Big);$$

$\mathcal{F}, \mathcal{F}'$ are i.o.w.s at time $t_2 - \frac{\varepsilon}{2}$, while $\mathcal{G}, \mathcal{G}'$ are i.o.w.s at time t_2. Moreover, since $\tau \mapsto \mathtt{f}_k^{\mathrm{eff}}(t)(\tau)$ is defined up to affine function, we can assume that

$$\frac{d\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\tau}\Big(\inf \Phi_k(t_2)(\mathcal{K})\Big) = \frac{d\mathtt{f}_k^{\mathrm{eff}}\big(t_2 - \frac{\varepsilon}{2}\big)}{d\tau}\Big(\inf \Phi_k\Big(t_2 - \frac{\varepsilon}{2}\Big)(\mathcal{K})\Big) = 0. \tag{3.63}$$

We divide now the proof of the second inequality in several steps.

Step 1. Define

$$\mathcal{H} := \mathcal{K} \cup \big\{z \in \mathcal{J} \cap \mathcal{W}_k(t_2) \mid z > \mathcal{K}\big\}, \qquad \mathcal{H}' := \mathcal{K}' \cup \big\{z \in \mathcal{J}' \cap \mathcal{W}_k(t_2) \mid z < \mathcal{K}'\big\}.$$

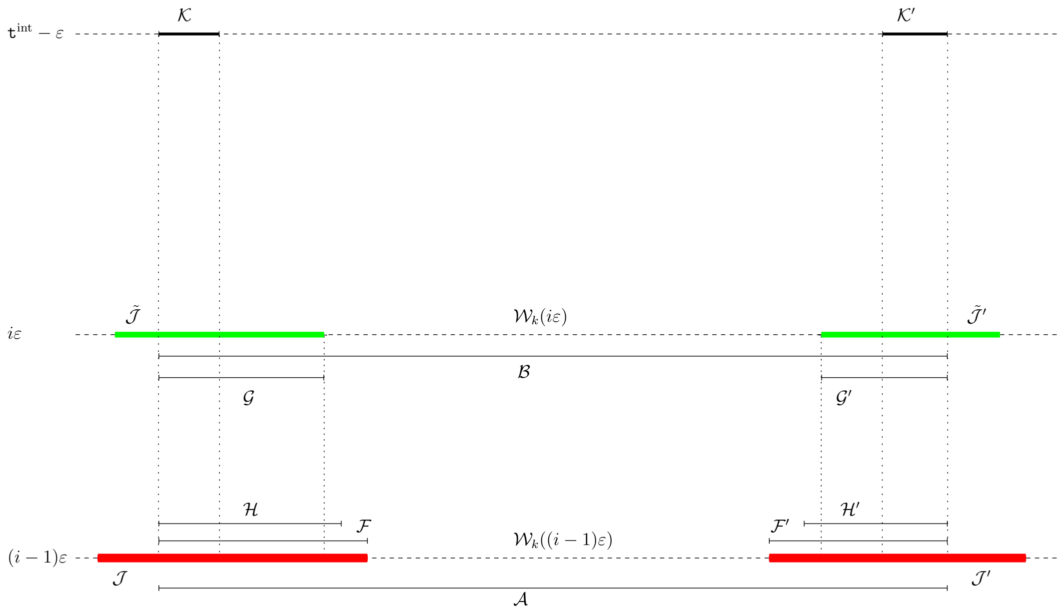

FIGURE 4. The various set used in the proof of (3.59b): in Step 2 pass from the waves in $\mathcal{F}, \mathcal{F}'$ to the waves $\mathcal{H}, \mathcal{H}'$ which survives at $t = t_2$; in Step 3 change the flux $\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right)$ to $\mathtt{f}_k^{\mathrm{eff}}(t_2)$ for the intervals $\mathcal{H}, \mathcal{H}'$; in Step 4 observe that $\mathcal{G}, \mathcal{G}'$ are shorter that $\mathcal{H}, \mathcal{H}'$ because of a splitting has occurred.

We now show that the sets $\mathcal{H}, \mathcal{H}'$ are i.o.w.s both at time t_2 and at time $t_2 - \frac{\varepsilon}{2}$ and

$$\mathcal{H} \subseteq \mathcal{J} \cap \mathcal{W}_k(t_2), \qquad \mathcal{H}' \subseteq \mathcal{J}' \cap \mathcal{W}_k(t_2).$$

Moreover also the sets

$$H_{t_2-\varepsilon/2} := \Phi_k\left(t_2 - \frac{\varepsilon}{2}\right)(\mathcal{H}), \qquad\qquad H_{t_2} := \Phi_k(t_2)(\mathcal{H}),$$
$$H'_{t_2-\varepsilon/2} := \Phi_k\left(t_2 - \frac{\varepsilon}{2}\right)(\mathcal{H}'), \qquad\qquad H'_{t_2} := \Phi_k(t_2)(\mathcal{H}'),$$

are intervals in $\mathbb{R}$.

Proof of Step 1. We prove only the statements related to $\mathcal{H}$, those related to $\mathcal{H}'$ being completely analogous. Clearly $\mathcal{H} \subseteq \mathcal{J} \cap \mathcal{W}_k(t_2)$. Moreover the set

$$\mathcal{M} := \left\{z \in \mathcal{W}_k(t_2) \;\middle|\; z \in \mathcal{K} \text{ or } z > \mathcal{K}\right\}$$

is clearly an i.o.w. at time t_2. Since we can write $\mathcal{H}$ as intersection of two i.o.w.s at time t_2 as

$$\mathcal{H} = \mathcal{M} \cap \big(\mathcal{J} \cap \mathcal{W}_k(t_2)\big),$$

it follows that also $\mathcal{H}$ is an i.o.w. at time t_2. Moreover, since $\mathcal{H} = \mathcal{H} \cap \mathcal{W}_k\left(t_2 - \frac{\varepsilon}{2}\right)$, by Proposition 3.52 and Corollary 3.26, Point (2), $\mathcal{H}$ is an i.o.w. also at time $t_2 - \frac{\varepsilon}{2}$. As an immediate consequence $H_{t_2-\varepsilon/2}$ and H_{t_2} are intervals in $\mathbb{R}$.

Step 2. We have

$$\left|\sigma^{\mathrm{rh}}\left(\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right), \mathcal{H}\right) - \sigma^{\mathrm{rh}}\left(\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right), \mathcal{F}\right)\right| \leq \mathcal{O}(1)\mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, m\varepsilon).$$

and

$$\left|\sigma^{\mathrm{rh}}\left(\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right), \mathcal{H}'\right) - \sigma^{\mathrm{rh}}\left(\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right), \mathcal{F}'\right)\right| \leq \mathcal{O}(1)\mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, m'\varepsilon).$$

Proof of Step 2. We prove only the first part of the statement, the second one being completely similar. Clearly $\mathcal{H} \subseteq \mathcal{F}$. Moreover, by Proposition 3.52, it follows that

$$\mathcal{F} \setminus \mathcal{H} \subseteq \left\{w \in \mathcal{W}_k\left(t_2 - \frac{\varepsilon}{2}\right) \setminus \mathcal{W}_k(t_2) \;\middle|\; \mathtt{x}(i\varepsilon, w) = m\varepsilon\right\}$$

and thus

$$\begin{aligned}
\mathcal{L}^1(F) - \mathcal{L}^1(H_{t_2-\varepsilon/2}) &= \mathcal{L}^1(\mathcal{F}) - \mathcal{L}^1(\mathcal{H}) \\
&= \mathcal{L}^1(\mathcal{F} \setminus \mathcal{H}) \\
&\leq \mathcal{L}^1\left(\left\{w \in \mathcal{W}_k\left(t_2 - \frac{\varepsilon}{2}\right) \setminus \mathcal{W}_k(t_2) \;\middle|\; \lim_{t\to t_2} \mathtt{x}(t, w) = m\varepsilon\right\}\right) \\
&\leq \mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, m\varepsilon).
\end{aligned}$$

Moreover, by Proposition 1.14,

$$\left|\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2 - \frac{\varepsilon}{2}, \mathcal{H}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2 - \frac{\varepsilon}{2}, \mathcal{F})\right| \leq \mathcal{L}^1(F) - \mathcal{L}^1(H_{t_2-\varepsilon/2}) \leq \mathcal{O}(1)\mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, m\varepsilon).$$

Step 3. It holds

$$\left|\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{H}\big) - \sigma^{\mathrm{rh}}\left(\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right), \mathcal{H}\right)\right| \leq \mathcal{O}(1) \sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon),$$

$$\left|\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{H}'\big) - \sigma^{\mathrm{rh}}\left(\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right), \mathcal{H}'\right)\right| \leq \mathcal{O}(1) \sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon).$$

Proof of Step 3. In this step, we prove only the second inequality and assume that $\mathcal{L}^1(\mathcal{H}) = \mathcal{L}^1(H_{t_2}) = \mathcal{L}^1(H_{t_2-\varepsilon/2}) > 0$, since the first inequality and the other cases can be treated similarly (and actually the computations are simpler).

We have

$$\begin{aligned}
&\left|\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{H}'\big) - \sigma^{\mathrm{rh}}\left(\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right), \mathcal{H}'\right)\right| \\
&\qquad = \left|\frac{1}{\mathcal{L}^1(H'_{t_2})} \int_{H'_{t_2}} \frac{d\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\varsigma}(\varsigma)d\varsigma - \frac{1}{\mathcal{L}^1(H'_{t_2-\varepsilon/2})} \int_{H'_{t_2-\varepsilon/2}} \frac{d\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right)}{d\tau}(\tau)d\tau\right| \\
&\text{(by (3.63))} = \left|\frac{1}{\mathcal{L}^1(H'_{t_2})} \int_{H'_{t_2}} \int_{\inf \Phi_k(t_2)(\mathcal{K})}^{\varsigma} \frac{d^2\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\xi^2}(\xi)d\xi d\varsigma\right. \\
&\qquad\qquad \left. - \frac{1}{\mathcal{L}^1(H'_{t_2-\varepsilon/2})} \int_{H'_{t_2-\varepsilon/2}} \int_{\inf \Phi_k(t_2-\varepsilon/2)(\mathcal{K})}^{\tau} \frac{d^2\mathtt{f}_k^{\mathrm{eff}}\left(t_2 - \frac{\varepsilon}{2}\right)}{d\eta^2}(\eta)d\eta d\tau\right|,
\end{aligned}$$

and, remembering that $\mathcal{L}^1(H'_{t_2}) = \mathcal{L}^1(H'_{t_2-\varepsilon/2})$ and integrating by parts,

$$\begin{aligned}
\dots &= \frac{1}{\mathcal{L}^1(H'_{t_2})}\bigg| \int_{\inf \Phi_k(t_2)(\mathcal{K})}^{\sup H'_{t_2}} \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\xi^2}(\xi)\Big(\sup H'_{t_2} - \max\big\{\xi, \inf H'_{t_2}\big\}\Big) d\xi \\
&\qquad\qquad - \int_{\inf \Phi_k(t_2-\varepsilon/2)(\mathcal{K})}^{\sup H'_{t_2-\varepsilon/2}} \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}\Big(t_2 - \frac{\varepsilon}{2}\Big)}{d\eta^2}(\eta) \\
&\qquad\qquad\qquad\qquad \cdot \Big(\sup H'_{t_2-\varepsilon/2} - \max\big\{\eta, \inf H'_{t_2-\varepsilon/2}\big\}\Big) d\eta \bigg| \\
&= \frac{1}{\mathcal{L}^1(H'_{t_2})}\bigg| \sum_{r=m}^{m'} \int_{[\inf \Phi_k(t_2)(\mathcal{K}), \sup H'_{t_2}]\cap K_r} \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\xi^2}(\xi)\Big(\sup H'_{t_2} - \max\big\{\xi, \inf H'_{t_2}\big\}\Big) d\xi \\
&\qquad - \sum_{r=m}^{m'} \int_{[\inf \Phi_k(t_2-\varepsilon/2)(\mathcal{K}), \sup H'_{t_2-\varepsilon/2}]\cap J_r} \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}\Big(t_2 - \frac{\varepsilon}{2}\Big)}{d\eta^2}(\eta) \\
&\qquad\qquad\qquad\qquad \cdot \Big(\sup H'_{t_2-\varepsilon/2} - \max\big\{\eta, \inf H'_{t_2-\varepsilon/2}\big\}\Big) d\eta \bigg|,
\end{aligned}$$

where K_r, J_r are defined in (3.53); using now that $\mathcal{L}^1(K_r \backslash T_r) = \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon)$, while $\mathcal{L}^1(J_r \backslash S_r) = \mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, r\varepsilon)$ we can proceed as

$$\begin{aligned}
\dots = \frac{1}{\mathcal{L}^1(H'_{t_2})}\bigg| & \sum_{r=m}^{m'} \mathcal{O}(1)\mathcal{L}^1(H_{t_2})\Big(\mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) + \mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, r\varepsilon)\Big) \\
& + \sum_{r=m}^{m'} \int_{[\inf \Phi_k(t_2)(\mathcal{K}), \sup H'_{t_2}]\cap T_r} \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\xi^2}(\xi)\Big(\sup H'_{t_2} - \max\big\{\xi, \inf H'_{t_2}\big\}\Big) d\xi \\
& - \sum_{r=m}^{m'} \int_{[\inf \Phi_k(t_2-\varepsilon/2)(\mathcal{K}), \sup H'_{t_2-\varepsilon/2}]\cap S_r} \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\eta^2}(\eta) \\
& \qquad\qquad\qquad \cdot \Big(\sup H'_{t_2-\varepsilon/2} - \max\big\{\eta, \inf H'_{t_2-\varepsilon/2}\big\}\Big) d\eta \bigg|,
\end{aligned}$$

and thus, using (3.62),

$$\begin{aligned}
\dots \leq{} & \frac{1}{\mathcal{L}^1(H'_{t_2})} \sum_{r=m}^{m'} \mathcal{O}(1) \mathcal{L}^1(H_{t_2}) \Big(\mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, m\varepsilon) + \mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, m\varepsilon) \Big) \\
& + \frac{1}{\mathcal{L}^1(H'_{t_2})} \Bigg| \sum_{r=m}^{m'} \int_{[\inf \Phi_k(t_2-\varepsilon/2)(\mathcal{K}), \sup H'_{t_2-\varepsilon/2}] \cap S_r} \Bigg[\frac{d^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\xi^2}(\Theta(\eta)) - \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}\big(t_2 - \frac{\varepsilon}{2}\big)}{d\eta^2}(\eta) \Bigg] \\
& \qquad \cdot \Big(\sup H'_{t_2-\varepsilon/2} - \max\big\{\eta, \inf H'_{t_2-\varepsilon/2}\big\} \Big) d\eta \Bigg| \\
\leq{} & \mathcal{O}(1) \sum_{r=m}^{m'} \Big(\mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) + \mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, r\varepsilon) \Big) + \sum_{r=m}^{m'} \int_{S_r} \Bigg| \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\xi^2}(\Theta(\eta)) - \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}\big(t_2 - \frac{\varepsilon}{2}\big)}{d\eta^2}(\eta) \Bigg| d\eta \\
\leq{} & \mathcal{O}(1) \sum_{r=m}^{m'} \Bigg(\mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) + \mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, r\varepsilon) + \Bigg\| \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\xi^2} \circ \Theta - \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}\big(t_2 - \frac{\varepsilon}{2}\big)}{d\eta^2} \Bigg\|_{L^1(S_r)} \Bigg),
\end{aligned}$$

and finally by Theorem 3.18 and Corollary 2.10

$$\begin{aligned}
& \Big| \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{H}') - \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2 - \frac{\varepsilon}{2}\Big), \mathcal{H}'\Big) \Big| \\
& \qquad \leq \mathcal{O}(1) \sum_{r=m}^{m'} \Bigg(\mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, r\varepsilon) + \mathtt{A}_k^{\mathrm{canc}}(i\varepsilon, r\varepsilon) + \Bigg\| \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\xi^2} \circ \Theta - \frac{d^2 \mathtt{f}_k^{\mathrm{eff}}\big(t_2 - \frac{\varepsilon}{2}\big)}{d\eta^2} \Bigg\|_{L^1(S_r)} \Bigg) \\
& \qquad \leq \mathcal{O}(1) \sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon).
\end{aligned}$$

Step 4. It holds

$$\Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{G}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{G}') \Big]^+ - \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{H}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{H}') \Big]^+ \leq 0.$$

Proof of Step 4. We want to use Proposition 1.15 with

$$g = \mathtt{f}_k^{\mathrm{eff}}(t_2), \quad [a,b] = \Phi_k(t_2)(\mathcal{J} \cap \mathcal{W}_k(t_2)), \quad \bar{u} = \sup \Phi_k(t_2)(\tilde{\mathcal{J}}), \quad u = \inf \Phi_k(t_2)(\mathcal{K}).$$

Indeed, by definition of the partition $\mathcal{P}(t_1, t_2, w, w')$ (Point (2a) at page 65), it holds

$$\operatorname*{conv}_{\Phi_k(t_2)(\mathcal{J} \cap \mathcal{W}_k(t_2))} \mathtt{f}_k^{\mathrm{eff}}(t_2)(\sup \Phi_k(t_2)(\tilde{\mathcal{J}})) = \mathtt{f}_k^{\mathrm{eff}}(t_2)(\sup \Phi_k(t_2)(\tilde{\mathcal{J}})),$$

i.e. $\operatorname*{conv}_{[a,b]} g(\bar{u}) = g(\bar{u})$.

We thus have

$$\begin{aligned}
\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{G}) &= \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}(t_2), \big[\inf \Phi_k(t_2)(\mathcal{K}), \sup \Phi_k(t_2)(\tilde{\mathcal{J}}) \big] \Big) \\
&\leq \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}(t_2), \big[\inf \Phi_k(t_2)(\mathcal{K}), \sup \Phi_k(t_2)(\mathcal{J} \cap \mathcal{W}_k(t_2)) \big] \Big) \\
&= \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{H}).
\end{aligned} \tag{3.64}$$

In a similar way one can prove that

$$\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2, \mathcal{G}') \geq \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2, \mathcal{H}'). \tag{3.65}$$

Using (3.64) and (3.65), one gets the conclusion.

Step 5. We can finally conclude the proof of (3.59b), showing that

$$\Delta\pi(\tau,\tau') \le \mathcal{O}(1)\sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, m\varepsilon).$$

Proof of Step 5. We can perform the following computation:

$$\begin{aligned}
&\Delta\pi(\tau,\tau')\\
&\quad= \pi\big(t_2,\Theta(\tau),\Theta(\tau')\big) - \pi\Big(t_2-\frac{\varepsilon}{2},\tau,\tau'\Big)\\
&\quad= \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{G}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{G}')\Big]^+\\
&\qquad\qquad - \bigg[\sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{F}\Big) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{F}')\bigg]^+\\
&\quad= \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{G}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{G}')\Big]^+ - \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{H}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{H}')\Big]^+\\
&\qquad + \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{H}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{H}')\Big]^+\\
&\qquad - \bigg[\sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{H}\Big) - \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{H}'\Big)\bigg]^+\\
&\qquad + \bigg[\sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{H}\Big) - \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{H}'\Big)\bigg]^+\\
&\qquad - \bigg[\sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{F}\Big) - \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{F}'\Big)\bigg]^+\\
&\quad\le \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{G}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{G}')\Big]^+ - \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{H}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{H}')\Big]^+\\
&\qquad + \bigg|\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{H}) - \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{H}\Big)\bigg|\\
&\qquad + \bigg|\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2),\mathcal{H}') - \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{H}'\Big)\bigg|\\
&\qquad + \bigg|\sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{H}\Big) - \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{F}\Big)\bigg|\\
&\qquad + \bigg|\sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{H}'\Big) - \sigma^{\mathrm{rh}}\Big(\mathtt{f}_k^{\mathrm{eff}}\Big(t_2-\frac{\varepsilon}{2}\Big),\mathcal{F}'\Big)\bigg|\\
&\quad\text{(by Steps 2, 3, 4 above)}\\
&\quad\le \mathcal{O}(1)\sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, m\varepsilon). \qquad \square
\end{aligned}$$

This concludes the proof of (3.59b).

Proof of (3.59c). The proof of (3.59c) is an immediate consequence of (3.59a) and (3.59b). First of all observe that, by definition of $d_k\big(t_1, t_2 - \frac{\varepsilon}{2}, t_3, w, w'\big)$, it holds

$$d_k\Big(t_1, t_2 - \frac{\varepsilon}{2}, t_3, w, w'\Big) \geq \Phi_k\big(t_2 - \frac{\varepsilon}{2}\big)(w') - \Phi_k\big(t_2 - \frac{\varepsilon}{2}\big)(w). \tag{3.66}$$

We thus have

$$\begin{aligned}
\Delta\mathfrak{q}_k(\tau,\tau') &= \mathfrak{q}_k\big(t_1,t_2,t_3,w,w'\big) - \mathfrak{q}_k\Big(t_1, t_2 - \frac{\varepsilon}{2}, t_3, w, w'\Big) \\
&= \frac{\pi_k\big(t_1,t_2,t_3,w,w'\big)}{d_k\big(t_1,t_2,t_3,w,w'\big)} - \frac{\pi_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big)}{d_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big)} \\
&= \pi_k\big(t_1,t_2,t_3,w,w'\big)\left(\frac{1}{d_k\big(t_1,t_2,t_3,w,w'\big)} - \frac{1}{d_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big)}\right) \\
&\qquad + \frac{1}{d_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big)} \\
&\qquad\qquad \cdot \left(\pi_k\big(t_1,t_2,t_3,w,w'\big) - \pi_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big)\right) \\
&\leq \frac{1}{d_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big)} \frac{\pi_k\big(t_1,t_2,t_3,w,w'\big)}{d_k\big(t_1,t_2,t_3,w,w'\big)} \big|\Delta d_k(\tau,\tau')\big| \\
&\qquad + \frac{1}{d_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big)} \Delta\pi_k(\tau,\tau') \\
&\leq \mathcal{O}(1)\frac{1}{d_k\Big(t_1,t_2-\frac{\varepsilon}{2},t_3,w,w'\Big)}\Big(\big|\Delta d_k(\tau,\tau')\big| + \Delta\pi_k(\tau,\tau')\Big) \\
\text{(by (3.66))} &\leq \mathcal{O}(1)\frac{1}{\Phi_k\big(t_2-\frac{\varepsilon}{2}\big)(w') - \Phi_k\big(t_2-\frac{\varepsilon}{2}\big)(w)}\Big(\big|\Delta d_k(\tau,\tau')\big| + \Delta\pi_k(\tau,\tau')\Big) \\
\text{(by (3.59a)-(3.59b))} &\leq \mathcal{O}(1)\frac{1}{\Phi_k\big(t_2-\frac{\varepsilon}{2}\big)(w') - \Phi_k\big(t_2-\frac{\varepsilon}{2}\big)(w)} \sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon).
\end{aligned}$$

□

We can now prove Theorem 3.61 and Theorem 3.62.

PROOF OF THEOREM 3.61. Fix $m < m'$, $w \in \mathcal{T}_m$, $w' \in \mathcal{T}_{m'}$. Observe first that, since $\mathtt{x}(i\varepsilon, w) = m\varepsilon < m'\varepsilon = \mathtt{x}(i\varepsilon, w')$, it must holds

$$\mathtt{t}^{\text{split}}(i\varepsilon, w, w') = \mathtt{t}^{\text{split}}\big((i-1)\varepsilon, w, w'\big). \tag{3.67}$$

Let now $\bar{t}$ be the time such that

$$\mathfrak{q}_k(i\varepsilon, i\varepsilon, w, w') := \sup_{\substack{t_3 \geq i\varepsilon \\ w,w' \in \mathcal{W}_k(t_3)}} \mathfrak{q}_k(i\varepsilon, i\varepsilon, t_3, w, w') = \mathfrak{q}_k(i\varepsilon, i\varepsilon, \bar{t}, w, w').$$

We have

$$\begin{aligned}
&\mathfrak{q}_k(i\varepsilon, w, w') - \mathfrak{q}\big((i-1)\varepsilon, w, w'\big) \\
&\qquad = \mathfrak{q}_k(i\varepsilon, i\varepsilon, w, w') - \mathfrak{q}\big((i-1)\varepsilon, (i-1)\varepsilon, w, w'\big) \\
&\qquad \leq \mathfrak{q}_k(i\varepsilon, i\varepsilon, \bar{t}, w, w') - \mathfrak{q}\big((i-1)\varepsilon, (i-1)\varepsilon, \bar{t}, w, w'\big) \\
&\text{(by Remark 3.56 and (3.67))} \\
&\qquad = \mathfrak{q}_k\big((i-1)\varepsilon, i\varepsilon, \bar{t}, w, w'\big) - \mathfrak{q}\big((i-1)\varepsilon, (i-1)\varepsilon, \bar{t}, w, w'\big) \\
&\qquad = \mathfrak{q}_k\big((i-1)\varepsilon, i\varepsilon, \bar{t}, w, w'\big) - \mathfrak{q}\Big((i-1)\varepsilon, \big(i-1/2\big)\varepsilon, \bar{t}, w, w'\Big) \qquad (3.68) \\
&\qquad\qquad + \mathfrak{q}\Big((i-1)\varepsilon, \big(i-1/2\big)\varepsilon, \bar{t}, w, w'\Big) \\
&\qquad\qquad - \mathfrak{q}\big((i-1)\varepsilon, (i-1)\varepsilon, \bar{t}, w, w'\big) \\
&\text{(by (3.59c) and using the fact that } \Phi_k((i-1)\varepsilon) = \Phi_k\big((i-1/2)\varepsilon\big)) \\
&\qquad \leq \mathcal{O}(1) \frac{\sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon)}{\Phi_k\big((i-1)\varepsilon\big)(w') - \Phi_k\big((i-1)\varepsilon\big)(w)}
\end{aligned}$$

To conclude the proof of Theorem 3.61, we can now use the change of variable

$$\Phi_k\big((i-1)\varepsilon\big) : \mathcal{W}_k\big((i-1)\varepsilon\big) \to \mathbf{I}\Big(V_k^+\big((i-1)\varepsilon\big)\Big),$$

whose properties are described in Proposition 3.31, as follows. Set, for simplicity, for any $m \in \mathbb{Z}$

$$T_m = \Phi_k\big((i-1)\varepsilon\big)\Big(\mathcal{T}_m\Big).$$

We thus have

$$\begin{aligned}
&\sum_{m<m'} \bigg\{ \iint_{\mathcal{T}_m \times \mathcal{T}_{m'}} \Big[\mathfrak{q}_k(i\varepsilon, w, w') - \mathfrak{q}_k\big((i-1)\varepsilon, w, w'\big)\Big] dw dw' \\
&\text{(by (3.68))} \leq \mathcal{O}(1) \sum_{m<m'} \iint_{\mathcal{T}_m \times \mathcal{T}_{m'}} \frac{\sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon)}{\Phi_k\big((i-1)\varepsilon\big)(w') - \Phi_k\big((i-1)\varepsilon\big)(w)} dw dw' \\
&\text{(changing variables)} = \mathcal{O}(1) \sum_{m<m'} \iint_{T_m \times T_{m'}} \frac{1}{\tau' - \tau} \sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon) d\tau d\tau' \\
&\qquad = \mathcal{O}(1) \Bigg[\sum_{r \in \mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon) \sum_{m=-\infty}^{r} \sum_{m'=r+1}^{+\infty} \iint_{T_m \times T_{m'}} \frac{1}{\tau' - \tau} d\tau d\tau' \\
&\qquad\qquad + \sum_{r \in \mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon) \sum_{m=-\infty}^{r-1} \iint_{T_m \times T_r} \frac{1}{\tau' - \tau} d\tau d\tau' \Bigg] \\
&\qquad \leq \mathcal{O}(1) \Bigg[\sum_{r \in \mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon) \int_0^{\sup T_r} \int_{\inf T_{r+1}}^{+\infty} \frac{1}{\tau' - \tau} d\tau d\tau \\
&\qquad\qquad + \sum_{r \in \mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon) \int_0^{\sup T_{r-1}} \int_{\inf T_r}^{+\infty} \frac{1}{\tau' - \tau} d\tau d\tau' \Bigg],
\end{aligned}$$

and since $\sup T_{r-1} < \inf T_r \leq \sup T_r < \inf R_{r+1}$ after an elementary integration by parts,

$$\begin{aligned} \dots &\leq \mathcal{O}(1) V_k\big((i-1)\varepsilon\big) \sum_{r\in\mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon) \\ \text{(by (2.18))} &\leq \mathcal{O}(1)\text{Tot.Var.}(u(0)) \sum_{r\in\mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon), \end{aligned}$$

thus concluding the proof of Theorem 3.61. □

Proof of Theorem 3.62. The proof of Theorem 3.62 is very similar to the proof of Theorem 3.61.

Fix m, $w \in \mathcal{J}_m^L$, $w' \in \mathcal{J}_m^R$. Observe first that

$$\mathtt{t}^{\text{split}}\Big(\big(i - \frac{1}{2}\big)\varepsilon, w, w'\Big) = \mathtt{t}^{\text{split}}\big((i-1)\varepsilon, w, w'\big). \tag{3.69}$$

Let now $\bar{t}$ be the time such that

$$\begin{aligned} \mathfrak{q}_k\Big(\big(i-\frac{1}{2}\big)\varepsilon, \big(i-\frac{1}{2}\big)\varepsilon, w, w'\Big) &:= \sup_{\substack{t_3 \geq (i-1/2)\varepsilon \\ w,w' \in \mathcal{W}_k(t_3)}} \mathfrak{q}_k\Big(\big(i-\frac{1}{2}\big)\varepsilon, \big(i-\frac{1}{2}\big)\varepsilon, t_3, w, w'\Big) \\ &= \mathfrak{q}_k\Big(\big(i-\frac{1}{2}\big)\varepsilon, \big(i-\frac{1}{2}\big)\varepsilon, \bar{t}, w, w'\Big). \end{aligned}$$

We have

$$\begin{aligned} &\mathfrak{q}_k\Big(\big(i-\frac{1}{2}\big)\varepsilon, w, w'\Big) - \mathfrak{q}\big((i-1)\varepsilon, w, w'\big) \\ &\qquad = \mathfrak{q}_k\Big(\big(i-\frac{1}{2}\big)\varepsilon, \big(i-\frac{1}{2}\big)\varepsilon, w, w'\Big) - \mathfrak{q}\big((i-1)\varepsilon, (i-1)\varepsilon, w, w'\big) \\ &\qquad \leq \mathfrak{q}_k\Big(\big(i-\frac{1}{2}\big)\varepsilon, \big(i-\frac{1}{2}\big)\varepsilon, \bar{t}, w, w'\Big) - \mathfrak{q}\big((i-1)\varepsilon, (i-1)\varepsilon, \bar{t}, w, w'\big) \\ \text{(by Remark 3.56 and (3.69))} &\qquad = \mathfrak{q}_k\Big((i-1)\varepsilon, \big(i-\frac{1}{2}\big)\varepsilon, \bar{t}, w, w'\Big) - \mathfrak{q}\big((i-1)\varepsilon, (i-1)\varepsilon, \bar{t}, w, w'\big) \\ \text{(by (3.59c))} &\qquad \leq \mathcal{O}(1) \frac{\mathtt{A}(i\varepsilon, m\varepsilon)}{\Phi_k\big((i-1)\varepsilon\big)(w') - \Phi_k\big((i-1)\varepsilon\big)(w)} \end{aligned} \tag{3.70}$$

As before, To conclude the proof of Theorem 3.62, we can now use the change of variable

$$\Phi_k\big((i-1)\varepsilon\big) : \mathcal{W}_k\big((i-1)\varepsilon\big) \to \mathbf{I}\Big(V_k^+\big((i-1)\varepsilon\big)\Big),$$

whose properties are described in Proposition 3.31, as follows.

Set, for simplicity, for any $m \in \mathbb{Z}$

$$J_m = \Phi_k\big(((i-1)\varepsilon\big)\Big(\mathcal{J}_m\Big), \qquad J_m^L = \Phi_k\big((i-1)\varepsilon\big)\Big(\mathcal{J}_m^L\Big), \qquad J_m^R = \Phi_k\big((i-1)\varepsilon\big)\Big(\mathcal{J}_m^R\Big), \tag{3.71}$$

We thus have

$$\begin{aligned} &\sum_{m\in\mathbb{Z}} \iint_{\mathcal{J}_m^L \times \mathcal{J}_m^R} \Big[\mathfrak{q}_k\Big(\big(i-\frac{1}{2}\big)\varepsilon\Big) - \mathfrak{q}_k\Big(\big(i-1\big)\varepsilon\Big)\Big] dw dw' \\ \text{(by (3.70))} &\leq \mathcal{O}(1) \sum_{m\in\mathbb{Z}} \iint_{\mathcal{J}_m^L \times \mathcal{J}_m^R} \frac{\mathtt{A}(i\varepsilon, m\varepsilon)}{\Phi_k\big((i-1)\varepsilon\big)(w') - \Phi_k\big((i-1)\varepsilon\big)(w)} dw dw' \\ \text{(changing variables)} &= \mathcal{O}(1) \sum_{m\in\mathbb{Z}} \iint_{J_m^L \times J_m^R} \frac{1}{\tau' - \tau} \mathtt{A}(i\varepsilon, m\varepsilon) d\tau d\tau' \end{aligned}$$

and since $\sup J_m^L < \inf J_m^R$ after an elementary integration by parts,

$$\dots \le \mathcal{O}(1) V_k\big((i-1)\varepsilon\big) \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon)$$

$$(\text{by } (2.18)) \le \mathcal{O}(1)\text{Tot.Var.}(u(0)) \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon),$$

thus concluding the proof of Theorem 3.62. □

3.5.5. Analysis of interacting waves. This section is devoted to conclude the proof of Theorem 3.58, showing that inequality (3.57) holds, i.e. estimating the (negative) term related to pairs of waves which are divided at time $(i-1)\varepsilon$ and are interacting at time $i\varepsilon$. In particular we will prove the following theorem:

THEOREM 3.64. *The following estimate holds:*

$$-\sum_{m\in\mathbb{Z}} \iint_{J_m^L\times J_m^R} \mathtt{q}_k((i-1)\varepsilon) d\tau d\tau' \le -\sum_{m\in\mathbb{Z}} \mathtt{A}_k^{\text{quadr}}(i\varepsilon, m\varepsilon) \tag{3.72}$$

We first separately prove the following proposition, which will be used also in Chapter 4, where the functional $\mathfrak{Q}$ is used to prove an estimate on the convergence rate of the Glimm scheme.

PROPOSITION 3.65. *Let $t_1 \le t_2$ be two fixed times. Let $\mathcal{J} = \mathcal{J}^L \cap \mathcal{J}^R$ be an i.o.w. at time t_2 which can be written as the union of two disjoint i.o.w. $\mathcal{J}^L$, $\mathcal{J}^R$, at time t_2, with $\mathcal{J}^L < \mathcal{J}^R$. Assume also that for any $(w, w') \in \mathcal{J}^L \times \mathcal{J}^R$*

(a) $\mathtt{p}(t_1, w, w')$ holds;
(b) $\mathcal{P}(t_1, t_2, w, w')$ can be restricted both to $\mathcal{J}^L$ and to $\mathcal{J}^R$.

Then, setting

$$J := \Phi_k(t_2)(\mathcal{J}), \quad J^L := \Phi_k(t_2)(\mathcal{J}^L), \quad J^R := \Phi_k(t_2)(\mathcal{J}^R),$$

it holds

$$\iint_{\mathcal{J}^L\times\mathcal{J}^R} \mathtt{q}_k(t_1, t_2, t_2, w, w') dw dw'$$
$$\ge \begin{cases} \Big\| D \operatorname*{conv}_J \mathtt{f}_k^{\text{eff}}(t_2) - \big(D \operatorname*{conv}_{J^L} \mathtt{f}_k^{\text{eff}}(t_2) \cup D \operatorname*{conv}_{J^R} \mathtt{f}_k^{\text{eff}}(t_2)\big)\Big\|_{L^1(J)}, & \text{if } \mathcal{S}(\mathcal{J}) = +1, \\ \Big\| D \operatorname*{conc}_J \mathtt{f}_k^{\text{eff}}(t_2) - \big(D \operatorname*{conc}_{J^L} \mathtt{f}_k^{\text{eff}}(t_2) \cup D \operatorname*{conc}_{J^R} \mathtt{f}_k^{\text{eff}}(t_2)\big)\Big\|_{L^1(J)}, & \text{if } \mathcal{S}(\mathcal{J}) = -1. \end{cases}$$

PROOF. We prove the proposition only in the case $\mathcal{S}(\mathcal{J}) = +1$, the negative case being completely similar. Set

$$\tau_M := \sup J^L = \inf J^R,$$

and

$$\tau_L := \max\Big\{\tau \in J^L \;\Big|\; \operatorname*{conv}_{J^L} \mathtt{f}_k^{\text{eff}}(t_2)(\tau) = \operatorname*{conv}_J \mathtt{f}_k^{\text{eff}}(t_2)(\tau)\Big\},$$
$$\tau_R := \min\Big\{\tau \in J^R \;\Big|\; \operatorname*{conv}_{J^R} \mathtt{f}_k^{\text{eff}}(t_2)(\tau) = \operatorname*{conv}_J \mathtt{f}_k^{\text{eff}}(t_2)(\tau)\Big\}.$$

W.l.o.g. we assume that $\tau_L < \tau_M < \tau_R$, otherwise there is nothing to prove.

It is not difficult to see that

$$\begin{aligned}
&\Big\| D \operatorname*{conv}_{J} \mathtt{f}_k^{\mathrm{eff}}(t_2) - \big(D \operatorname*{conv}_{J^L} \mathtt{f}_k^{\mathrm{eff}}(t_2) \cup D \operatorname*{conv}_{J^R} \mathtt{f}_k^{\mathrm{eff}}(t_2)\big) \Big\|_{L^1(J)} \\
&\quad = \frac{1}{\tau_R - \tau_L} \Big[\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), (\tau_L, \tau_M]\big) - \sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), (\tau_M, \tau_R]\big) \Big] \mathcal{L}^2\big((\tau_L, \tau_M] \times (\tau_M, \tau_R]\big).
\end{aligned}$$

We thus have to prove that

$$\begin{aligned}
&\frac{1}{\tau_R - \tau_L} \Big[\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), (\tau_L, \tau_M]\big) - \sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), (\tau_M, \tau_R]\big) \Big] \mathcal{L}^2\big((\tau_L, \tau_M] \times (\tau_M, \tau_R]\big) \\
&\qquad \leq \int_{\tau_L}^{\tau_M} \int_{\tau_M}^{\tau_R} \mathfrak{q}(t_1, t_2, t_2, w, w') dw dw'.
\end{aligned} \tag{3.73}$$

Let

$$\mathcal{L} := \Phi_k(t_2)^{-1}\big((\tau_L, \tau_M]\big), \qquad \mathcal{R} := \Phi_k(t_2)^{-1}\big((\tau_M, \tau_R]\big).$$

Observe that, by the assumption (b) in the statement of the proposition and by Remark 3.49, for any $(w, w') \in \mathcal{J}^L \times \mathcal{J}^R$, the partition $\mathcal{P}(t_1, t_2, w, w')$ can be restricted to both $\mathcal{L}$ and $\mathcal{R}$. We can thus assume w.l.o.g. that $\mathcal{I}(t_1, t_2, w, w') \subseteq \mathcal{L} \cup \mathcal{R}$ for any $(w, w') \in \mathcal{L} \times \mathcal{R}$. Therefore,

$$d(t_1, t_2, t_2, w, w') \leq \tau_R - \tau_L.$$

Hence (3.73) will follow if we prove that

$$\begin{aligned}
&\Big[\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), (\tau_L, \tau_M]\big) - \sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2), (\tau_M, \tau_R]\big) \Big] \mathcal{L}^2\big((\tau_L, \tau_M] \times (\tau_M, \tau_R]\big) \\
&\qquad \leq \int_{\Phi_k(t_2)^{-1}((\tau_L, \tau_M])} \int_{\Phi_k(t_2)^{-1}((\tau_M, \tau_R])} \pi(t_1, t_2, t_2, w, w') dw dw'.
\end{aligned} \tag{3.74}$$

We will identify waves through the equivalence relation $\bowtie$ introduced in (3.39): for any couple of waves $w, w' \in \mathcal{L} \cup \mathcal{R}$, set $w \bowtie w'$ if and only if

$$\mathtt{t}^{\mathrm{cr}}(w) = \mathtt{t}^{\mathrm{cr}}(w') \text{ and } \mathtt{x}(t, w) = \mathtt{x}(t, w') \quad \text{ for any } t \in \Big[\mathtt{t}^{\mathrm{cr}}(w), i\varepsilon\Big).$$

As observed in Lemma 3.41, the sets

$$\widehat{\mathcal{L}} := \mathcal{L} \,/\, \bowtie, \qquad \widehat{\mathcal{R}} := \mathcal{R} \,/\, \bowtie$$

are finite and totally ordered by the order $\leq$ on $\mathcal{W}_k^+(t_2)$. Moreover for any $\xi \in \widehat{\mathcal{L}}$, $\xi' \in \widehat{\mathcal{R}}$, let $w \in \xi$, $w' \in \xi'$ and set

$$\mathcal{I}(t_1, t_2, \xi, \xi') := \mathcal{I}(t_1, t_2, w, w'), \qquad \mathcal{P}(t_1, t_2, \xi, \xi') := \mathcal{P}(t_1, t_2, w, w'),$$

and

$$\widehat{\mathcal{I}}(t_1, t_2, \xi, \xi') := \mathcal{I}(t_1, t_2, \xi, \xi') \,/\, \bowtie.$$

The above definitions are well posed thanks to Lemma 3.50.

Now we partition the rectangle $\widehat{\mathcal{L}} \times \widehat{\mathcal{R}}$ in sub-rectangles, as follows. For any rectangle $\widehat{\mathcal{C}} := \widehat{\mathcal{L}}_{\mathcal{C}} \times \widehat{\mathcal{R}}_{\mathcal{C}} \subseteq \widehat{\mathcal{L}} \times \widehat{\mathcal{R}}$, define (see Figure 5)

$$\Pi_0(\widehat{\mathcal{C}}) := \begin{cases} \emptyset, & \widehat{\mathcal{C}} = \emptyset, \\ \Big[\widehat{\mathcal{L}}_{\mathcal{C}} \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big] \times \Big[\widehat{\mathcal{R}}_{\mathcal{C}} \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big], & \widehat{\mathcal{C}} \neq \emptyset, \end{cases}$$

$$\Pi_1(\widehat{\mathcal{C}}) := \begin{cases} \emptyset, & \widehat{\mathcal{C}} = \emptyset, \\ \Big[\widehat{\mathcal{L}}_{\mathcal{C}} \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big] \times \Big[\widehat{\mathcal{R}}_{\mathcal{C}} \setminus \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big], & \widehat{\mathcal{C}} \neq \emptyset, \end{cases}$$

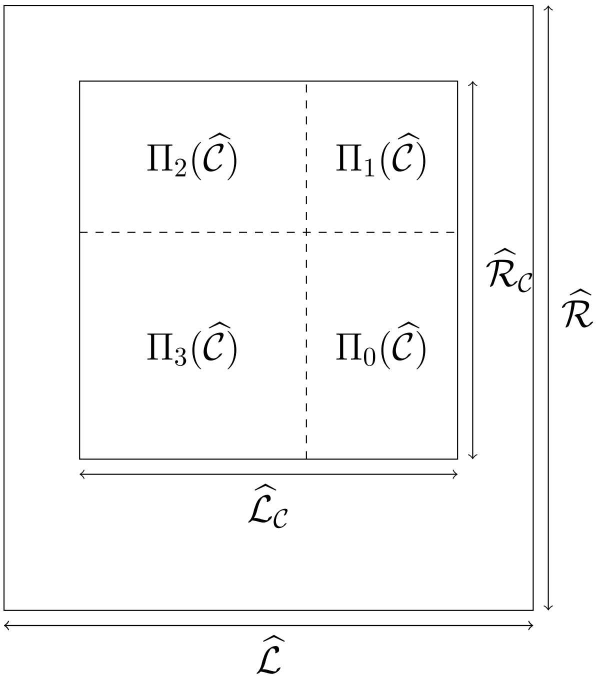

FIGURE 5. Partition of $\widehat{C} := \widehat{\mathcal{L}}_{\mathcal{C}} \times \widehat{\mathcal{R}}_{\mathcal{C}}$.

$$\Pi_2(\widehat{\mathcal{C}}) := \begin{cases} \emptyset, & \widehat{\mathcal{C}} = \emptyset, \\ \Big[\widehat{\mathcal{L}}_{\mathcal{C}} \setminus \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big] \times \Big[\widehat{\mathcal{R}}_{\mathcal{C}} \setminus \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big], & \widehat{\mathcal{C}} \neq \emptyset, \end{cases}$$

$$\Pi_3(\widehat{\mathcal{C}}) := \begin{cases} \emptyset, & \widehat{\mathcal{C}} = \emptyset, \\ \Big[\widehat{\mathcal{L}}_{\mathcal{C}} \setminus \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big] \times \Big[\widehat{\mathcal{R}}_{\mathcal{C}} \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big], & \widehat{\mathcal{C}} \neq \emptyset, \end{cases}$$

Clearly $\Big\{\Pi_0(\widehat{\mathcal{C}}), \Pi_1(\widehat{\mathcal{C}}), \Pi_2(\widehat{\mathcal{C}}), \Pi_3(\widehat{\mathcal{C}})\Big\}$ is a disjoint partition of $\widehat{\mathcal{C}}$.

For any set A, denote by $A^{<\mathbb{N}}$ the set of all finite sequences taking values in A. We assume that $\emptyset \in A^{<\mathbb{N}}$, called the *empty sequence*. There is a natural ordering $\trianglelefteq$ on $A^{<\mathbb{N}}$: given $\alpha, \beta \in A^{<\mathbb{N}}$,

$$\alpha \trianglelefteq \beta \quad \Longleftrightarrow \quad \beta \text{ is obtained from } \alpha \text{ by adding a finite sequence.}$$

A subset $D \subseteq A^{<\mathbb{N}}$ is called a *tree* if for any $\alpha, \beta \in A^{<\mathbb{N}}$, $\alpha \trianglelefteq \beta$, if $\beta \in D$, then $\alpha \in D$.

Define a map $\widehat{\Psi} : \{0,1,2,3\}^{<\mathbb{N}} \longrightarrow 2^{\widehat{\mathcal{L}} \times \widehat{\mathcal{R}}}$, by setting

$$\widehat{\Psi}_\alpha = \begin{cases} \widehat{\mathcal{L}} \times \widehat{\mathcal{R}}, & \text{if } \alpha = \emptyset, \\ \Pi_{a_n} \circ \cdots \circ \Pi_{a_1}(\widehat{\mathcal{L}} \times \widehat{\mathcal{R}}), & \text{if } \alpha = (a_1, \dots, a_n) \in \{0,1,2,3\}^{<\mathbb{N}} \setminus \{\emptyset\}. \end{cases}$$

For $\alpha \in \{0,1,2,3\}^{<\mathbb{N}}$, let $\widehat{\mathcal{L}}_\alpha, \widehat{\mathcal{R}}_\alpha$ be defined by the relation $\widehat{\Psi}_\alpha = \widehat{\mathcal{L}}_\alpha \times \widehat{\mathcal{R}}_\alpha$. Define a tree D in $\{0,1,2,3\}^{<\mathbb{N}}$ setting

$$D := \{\emptyset\} \cup \Big\{\alpha = (a_1, \dots, a_n) \in \{0,1,2,3\}^{<\mathbb{N}} \;\Big|\; n \in \mathbb{N},\ \widehat{\Pi}_\alpha \neq \emptyset,\ a_k \neq 0 \text{ for } k = 1, \dots, n-1\Big\}.$$

See Figure 6.

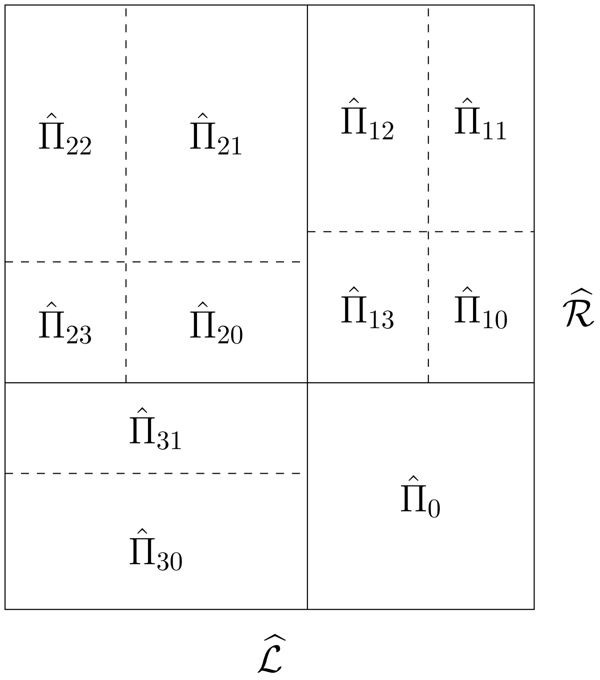

FIGURE 6. Partition of $\mathcal{L} \times \mathcal{R}$ using the tree D.

Since $\Pi_0(\Pi_0(\widehat{\mathcal{C}})) = \Pi_0(\widehat{\mathcal{C}})$ for any $\widehat{\mathcal{C}} \subseteq \widehat{\mathcal{L}} \times \widehat{\mathcal{R}}$, this implies, together with the fact that $\widehat{\mathcal{L}} \times \widehat{\mathcal{R}}$ is a finite set, that D is a finite tree.

For any $\alpha \in D$, set

$$\mathcal{L}_\alpha := \bigcup_{\xi \in \widehat{\mathcal{L}}_\alpha} \xi, \qquad \mathcal{R}_\alpha := \bigcup_{\xi' \in \widehat{\mathcal{R}}_\alpha} \xi',$$
$$L_\alpha := \Phi_k(t_2)(\mathcal{L}_\alpha), \qquad R_\alpha := \Phi_k(t_2)(\mathcal{R}_\alpha).$$

The idea of the proof is to show that, for each $\alpha \in D$, on the rectangle $L_\alpha \times R_\alpha$ it holds

$$\begin{aligned} &\big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), L_\alpha) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), R_\alpha)\big]\mathcal{L}^2(L_\alpha \times R_\alpha) \\ &\qquad \leq \int_{\Phi_k(t_2)^{-1}(L_\alpha) \times \Phi_k(t_2)^{-1}(R_\alpha)} \pi(t_1, t_2, t_2, w, w') dw dw'. \end{aligned} \tag{3.75}$$

The conclusion will follow just considering that $\emptyset \in D$ and $L_\emptyset = (\tau_L, \tau_M]$, $R_\emptyset = (\tau_M, \tau_R]$.

We now need the following two lemmas.

LEMMA 3.66. *For any $\beta \in D$, the partition $\mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ of the characteristic interval $\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ can be restricted to*

$$\mathcal{L}_\beta \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$$

and to

$$\mathcal{R}_\beta \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta).$$

PROOF. Let us prove only the first part of the statement, the second one being completely similar. We will show by induction the following stronger claim:
for each $\gamma \trianglelefteq \beta$, the partition $\mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ of the interval $\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ can be restricted to $\mathcal{L}_\gamma \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$.

For $\gamma = \emptyset$, by definition $\mathcal{L}_\emptyset = \mathcal{L}$ and thus the proof follows from assumption (b) in the statement of the proposition and Remark 3.49. Thus assume the claim is true for some $\gamma \lhd \beta$ and let us prove it for γa, with $a \in \{0,1,2,3\}$.

If $a = 0,1$, by definition it holds

$$\mathcal{L}_{\gamma a} = \mathcal{L}_\gamma \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\gamma, \min \widehat{\mathcal{R}}_\gamma).$$

Hence

$$\mathcal{L}_{\gamma a} \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta) = \mathcal{L}_\gamma \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\gamma, \min \widehat{\mathcal{R}}_\gamma) \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta).$$

By inductive assumption, the partition $\mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ of $\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ can be restricted to $\mathcal{L}_\gamma \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$, while, since $\gamma \lhd \beta$,

$$\max \widehat{\mathcal{L}}_\beta \leq \max \widehat{\mathcal{L}}_\gamma \leq \min \widehat{\mathcal{R}}_\gamma \leq \min \widehat{\mathcal{R}}_\beta$$

and therefore, by Proposition 3.54, the partition $\mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ can be restricted also to $\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\gamma, \min \widehat{\mathcal{R}}_\gamma) \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$, and thus we are done.

If $a = 2,3$, by definition it holds

$$\mathcal{L}_{\gamma a} = \mathcal{L}_\gamma \setminus \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\gamma, \min \widehat{\mathcal{R}}_\gamma).$$

Hence

$$\begin{aligned}
&\mathcal{L}_{\gamma a} \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta) \\
&\quad = \Big(\mathcal{L}_\gamma \setminus \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\gamma, \min \widehat{\mathcal{R}}_\gamma)\Big) \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta) \\
&\quad = \Big(\mathcal{L}_\gamma \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)\Big) \cap \Big(\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta) \setminus \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\gamma, \min \widehat{\mathcal{R}}_\gamma)\Big).
\end{aligned}$$

As in the case $a = 0,1$, by inductive assumption, the partition $\mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ of the interval $\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ can be restricted to $\mathcal{L}_\gamma \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$, while, as before, by Proposition 3.54 using $\gamma \lhd \beta$, it can be restricted also to $\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta) \setminus \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\gamma, \min \widehat{\mathcal{R}}_\beta)$, and thus we are done also in this case. □

LEMMA 3.67. *For each $\alpha = (a_1, \dots a_n) \in D$, if $a_n = 0$, then it holds*

$$\begin{aligned}
&\big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), L_\alpha) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), R_\alpha)\big]\mathcal{L}^2(L_\alpha \times R_\alpha) \\
&\qquad\qquad \leq \iint_{\Phi_k(t_2)^{-1}(L_\alpha) \times \Phi_k(t_2)^{-1}(R_\alpha)} \pi_k(t_1, t_2, t_2, w, w') dw dw'.
\end{aligned}$$

PROOF. Set $\beta := (a_1, \dots, a_{n-1})$. Since $a_n = 0$, then

$$\widehat{\Psi}_\alpha = \Pi_0(\widehat{\Psi}_\beta) = \Big(\widehat{\mathcal{L}}_\beta \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)\Big) \times \Big(\widehat{\mathcal{R}}_\beta \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)\Big),$$

and thus

$$\mathcal{L}_\alpha = \mathcal{L}_\beta \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta), \qquad \mathcal{R}_\alpha = \mathcal{R}_\beta \cap \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta).$$

Observe first that if one between $\max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta$ does not belong to $\mathcal{W}_k(t_1)$ or they both belong to $\mathcal{W}_k(t_1)$ but they have never interacted at time t_1, then by Proposition 3.55, Point (3), either one between w, w' does not belong to $\mathcal{W}_k(t_1)$ or $w, w' \in \mathcal{W}_k(t_1)$ but they have never interacted at time t_1. In particular

$$\pi_k(t_1, t_2, t_2, w, w') = \left[\frac{d\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\tau}(\tau')\right]^+$$

(by (3.46) and the fact that $\mathcal{K}, \mathcal{K}'$ in (3.76) are singletons) and thus the conclusion is trivial.

Assume then that $\max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta \in \mathcal{W}_K(t_1)$, are divided at time t_1 and have already interacted at time t_1. Consider the partition $\mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ of the interval $\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ and set

$$\mathbf{P} := \Big\{ \Phi_k(t_2)(\mathcal{J}) \;\Big|\; \mathcal{J} \in \mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta) \Big\}.$$

By definition of the partition in Section 3.4.3, the elements of $\mathbf{P}$ are intervals in $\mathbb{R}$, possibly singletons. Clearly the non-singleton intervals in $\mathbf{P}$ are at most countable; moreover by Lemma 3.66, the partition $\mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$ can be restricted both to $\mathcal{L}_\alpha$ and to $\mathcal{R}_\alpha$; hence, denoting by $\{U_r\}_{r\in\mathbb{N}}$ the non-singleton elements of $\mathbf{P}$ contained in L_α and by $\{V_{r'}\}_{r'\in\mathbb{N}}$ the non-singleton elements of $\mathbf{P}$ contained in R_α, we can write L_α, R_α as

$$\begin{aligned}
L_\alpha = \Phi_k(t_2)(\mathcal{L}_\alpha) &= \Big(\bigcup_{r\in\mathbb{N}} U_r \Big) \cup \Big(L_\alpha \setminus \bigcup_{r\in\mathbb{N}} U_r \Big), \\
R_\alpha = \Phi_k(t_2)(\mathcal{R}_\alpha) &= \Big(\bigcup_{r'\in\mathbb{N}} V_{r'} \Big) \cup \Big(R_\alpha \setminus \bigcup_{r'\in\mathbb{N}} V_{r'} \Big);
\end{aligned}$$

set also, for shortness:

$$U := \bigcup_{r\in\mathbb{N}} U_r, \qquad V := \bigcup_{r'\in\mathbb{N}} V_{r'}.$$

Now observe that for $(\tau, \tau') \in L_\alpha \times R_\alpha$, setting

$$w := \Phi_k(t_2)^{-1}(\tau), \qquad w' := \Phi_k(t_2)^{-1}(\tau'),$$

if $\mathcal{K}, \mathcal{K}' \in \mathcal{P}(t_1, t_2, w, w')$ are the elements of the partition containing w, w' respectively, then, by definition of π_k,

$$\pi_k(t_1, t_2, t_2, w, w') = \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{K}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{K}') \Big]^+. \tag{3.76}$$

Moreover it holds:

$$
\begin{aligned}
\Big[\sigma^{\mathrm{rh}}&(\mathtt{f}^{\mathrm{eff}}_k(t_2), L_\alpha) - \sigma^{\mathrm{rh}}(\mathtt{f}^{\mathrm{eff}}_k(t_2), R_\alpha)\Big]\mathcal{L}^2(L_\alpha \times R_\alpha) \\
&= \iint_{L_\alpha \times R_\alpha} \bigg[\frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau' \\
&= \sum_{r,r' \in \mathbb{N}} \iint_{U_r \times V_{r'}} \bigg[\frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau' \\
&\qquad + \sum_{r \in \mathbb{N}} \iint_{U_r \times (R_\alpha \setminus V)} \bigg[\frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau' \\
&\qquad + \sum_{r' \in \mathbb{N}} \iint_{(L_\alpha \setminus U) \times V_{r'}} \bigg[\frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau' \\
&\qquad + \iint_{(L_\alpha \setminus U) \times (R_\alpha \setminus V)} \bigg[\frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau' \\
&= \sum_{r,r' \in \mathbb{N}} \mathcal{L}^2(U_r \times V_{r'}) \Big[\sigma^{\mathrm{rh}}(\mathtt{f}^{\mathrm{eff}}_k(t_2), U_r) - \sigma^{\mathrm{rh}}(\mathtt{f}^{\mathrm{eff}}_k(t_2), V_{r'})\Big] \\
&\qquad + \sum_{r \in \mathbb{N}} \mathcal{L}^1(U_r) \int_{R_\alpha \setminus V} \bigg[\sigma^{\mathrm{rh}}(\mathtt{f}^{\mathrm{eff}}_k(t_2), U_r) - \frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau')\bigg] d\tau' \\
&\qquad + \sum_{r' \in \mathbb{N}} \mathcal{L}^1(V_{r'}) \int_{L_\alpha \setminus U} \bigg[\frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau) - \sigma^{\mathrm{rh}}(\mathtt{f}^{\mathrm{eff}}_k(t_2), V_{r'})\bigg] d\tau \\
&\qquad + \iint_{(L_\alpha \setminus U) \times (R_\alpha \setminus V)} \bigg[\frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau'.
\end{aligned} \tag{3.77}
$$

Now, if $\max \widehat{\mathcal{L}}_\beta$, $\min \widehat{\mathcal{R}}_\beta$ have never interacted at time $(i-1)\varepsilon$, then, by definition of the partition, $U = \emptyset$ and $V = \emptyset$. Therefore Proposition 3.55, Point (3), also $\mathcal{P}(t_1, t_2, w, w')$ is made by singletons for any $w, w' \in \mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$, and thus

$$
\begin{aligned}
\Big[\sigma^{\mathrm{rh}}&(\mathtt{f}^{\mathrm{eff}}_k(t_2), L_\alpha) - \sigma^{\mathrm{rh}}(\mathtt{f}^{\mathrm{eff}}_k(t_2), R_\alpha)\Big]\mathcal{L}^2(L_\alpha \times R_\alpha) \\
&\le \iint_{(L_\alpha) \times (R_\alpha)} \bigg[\frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\mathrm{eff}}_k(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau' \\
&= \iint_{(L_\alpha) \times (R_\alpha)} \pi_k(t_1, t_2, t_2, w, w') dw dw'.
\end{aligned}
$$

On the other hand, if if $\max \widehat{\mathcal{L}}_\beta$, $\min \widehat{\mathcal{R}}_\beta$ have already interacted at time $(i-1)\varepsilon$, then

(1) It holds:

$$
\begin{aligned}
\mathcal{L}^2(U_r \times V_{r'}) &\Big[\sigma^{\mathrm{rh}}(\mathtt{f}^{\mathrm{eff}}_k(t_2), U_r) - \sigma^{\mathrm{rh}}(\mathtt{f}^{\mathrm{eff}}_k(t_2), V_{r'})\Big] \\
&\le \iint_{\Phi_k(t_2)^{-1}(U_r) \times \Phi_k(t_2)^{-1}(V_{r'})} \pi_k(t_1, t_2, t_2, w, w') dw dw'.
\end{aligned} \tag{3.78}
$$

Indeed, if $(w, w') \in \Phi_k(t_2)^{-1}(U_r) \times \Phi_k(t_2)^{-1}(V_{r'})$, then, by the definition of the partition, $\mathtt{t}^{\mathrm{cr}}(w), \mathtt{t}^{\mathrm{cr}}(w') \le \mathtt{t}^{\mathrm{split}}(t_1, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta)$, and thus by Proposition 3.55, Point (1), w, w' and have already interacted at time t_1 and

$$
\mathcal{I}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta) = \mathcal{I}(t_1, t_2, w, w'), \ \mathcal{P}(t_1, t_2, \max \widehat{\mathcal{L}}_\beta, \min \widehat{\mathcal{R}}_\beta) = \mathcal{P}(t_1, t_2, w, w').
$$

Therefore

$$\pi_k(t_1,t_2,t_2,w,w') = \Big[\sigma^{\rm rh}(\mathtt{f}_k^{\rm eff}(t_2),U_r) - \sigma^{\rm rh}(\mathtt{f}_k^{\rm eff}(t_2),V_{r'})\Big]^+,$$

which implies (3.78)

(2) It holds

$$\begin{aligned}\mathcal{L}^1(U_r)\int_{R_\alpha\setminus V}&\bigg[\sigma^{\rm rh}(\mathtt{f}_k^{\rm eff}(t_2),U_r) - \frac{d\mathtt{f}_k^{\rm eff}(t_2)}{d\tau}(\tau')\bigg]d\tau' \\ &\leq \iint_{\Phi_k(t_2)^{-1}(U_r)\times\Phi_k(t_2)^{-1}(R_\alpha\setminus V)} \pi_k(t_1,t_2,t_2,w,w')dwdw'.\end{aligned} \tag{3.79}$$

Indeed, if for any $(w,w')\in U_r\times R_\alpha\setminus V$, we have $\mathtt{t}^{\rm cr}(w)\leq \mathtt{t}^{\rm split}(t_1,\max\widehat{\mathcal{L}}_\beta,\min\widehat{\mathcal{R}}_\beta)$. Now, if also $\mathtt{t}^{\rm cr}(w')\leq \mathtt{t}^{\rm split}(t_1,\max\widehat{\mathcal{L}}_\beta,\min\widehat{\mathcal{R}}_\beta)$, then, as before, by Proposition 3.55, Point (1),

$$\pi_k(t_1,t_2,t_2,w,w') = \bigg[\sigma^{\rm rh}(\mathtt{f}_k^{\rm eff}(t_2),U_r) - \frac{d\mathtt{f}_k^{\rm eff}(t_2)}{d\tau}(\tau')\bigg]^+,$$

which implies (3.79). On the other side, if $\mathtt{t}^{\rm cr}(w')> \mathtt{t}^{\rm split}(t_1,\max\widehat{\mathcal{L}}_\beta,\min\widehat{\mathcal{R}}_\beta)$, then by Proposition 3.55, Point (2), we have

$$\pi_k(t_1,t_2,t_2,w,w') = \bigg[\frac{d\mathtt{f}_k^{\rm eff}(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}_k^{\rm eff}(t_2)}{d\tau}(\tau')\bigg]^+,$$

which implies (3.79).

(3) Similarly to the previous point,

$$\begin{aligned}\mathcal{L}^1(V_{r'})\int_{L_\alpha\setminus U}&\bigg[\frac{d\mathtt{f}_k^{\rm eff}(t_2)}{d\tau}(\tau) - \sigma^{\rm rh}(\mathtt{f}_k^{\rm eff}(t_2),V_{r'})\bigg]d\tau \\ &\leq \iint_{\Phi_k(t_2)^{-1}(L_\alpha\setminus U)\times\Phi_k(t_2)^{-1}(V_{r'})} \pi_k(t_1,t_2,t_2,w,w')dwdw'.\end{aligned} \tag{3.80}$$

(4) Finally, again with a similar analysis,

$$\begin{aligned}\iint_{\Phi_k(t_2)^{-1}(L_\alpha\setminus U)\times\Phi_k(t_2)^{-1}(R_\alpha\setminus V)}&\bigg[\frac{d\mathtt{f}_k^{\rm eff}(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}_k^{\rm eff}(t_2)}{d\tau}(\tau')\bigg]d\tau d\tau' \\ &\leq \iint_{(L_\alpha\setminus U)\times(R_\alpha\setminus V)} \pi_k(t_1,t_2,t_2,w,w')dwdw'.\end{aligned} \tag{3.81}$$

It is now clear that (3.77) together with (3.78), (3.79), (3.80), (3.81) implies the thesis. ☐

Conclusion of the proof of Proposition 3.65. In the previous lemma we proved inequality (3.75) for the elements $\alpha\in D$ of the tree whose last component is equal to 0. Now we use this fact to prove (3.75) for any $\alpha\in D$. We proceed by (inverse) induction on the tree.

If α is a leaf of the tree, then, by definition, the last component of α is equal to zero, and thus Lemma 3.67 applies.

If α is not a leaf, then

$$\widehat{\Psi}_\alpha = \widehat{\Psi}_{\alpha 0}\cup\widehat{\Psi}_{\alpha 1}\cup\widehat{\Psi}_{\alpha 2}\cup\widehat{\Psi}_{\alpha 3}$$

and thus

$$L_\alpha\times R_\alpha = \Big(L_{\alpha 0}\times R_{\alpha 0}\Big)\cup\Big(L_{\alpha 1}\times R_{\alpha 1}\Big)\cup\Big(L_{\alpha 2}\times R_{\alpha 2}\Big)\cup\Big(L_{\alpha 3}\times R_{\alpha 3}\Big).$$

The estimate (3.75) holds on $L_{\alpha 0} \times R_{\alpha 0}$ by Lemma 3.67, while it holds on $L_{\alpha a} \times R_{\alpha a}$, $a = 1, 2, 3$, by inductive assumption. Hence we can write

$$\begin{aligned}
\big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), L_\alpha) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), R_\alpha)\big]\mathcal{L}^2(L_\alpha \times R_\alpha)& \\
&= \iint_{L_\alpha \times R_\alpha} \bigg[\frac{d\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau' \\
&= \sum_{a=0}^{3} \iint_{L_{\alpha a} \times R_{\alpha a}} \bigg[\frac{d\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}_k^{\mathrm{eff}}(t_2)}{d\tau}(\tau')\bigg] d\tau d\tau' \\
&= \sum_{a=0}^{3} \big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), L_{\alpha a}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), R_{\alpha a})\big]\mathcal{L}^2(L_{\alpha a} \times R_{\alpha a}) \\
&\le \sum_{a=0}^{3} \iint_{\Phi_k(t_2)^{-1}(L_{\alpha a}) \times \Phi_k(t_2)^{-1}(R_{\alpha a})} \pi(t_1, t_2, t_2, w, w') dw dw' \\
&= \iint_{\Phi_k(t_2)^{-1}(L_\alpha) \times \Phi_k(t_2)^{-1}(R_\alpha)} \pi(t_1, t_2, t_2, w, w') dw dw'.
\end{aligned}$$

As already observed, for $\alpha = \emptyset$, we get inequality (3.74), thus concluding the proof of the proposition. $\square$

We can finally use Proposition 3.65 to prove Theorem 3.64, thus concluding the proof of Theorem 3.58 and, therefore, also the proof of Theorem A.

Proof of Theorem 3.64. Fix $m \in \mathbb{Z}$. Let J_m, J_m^L, J_m^R as in (3.71). By definition the effective flux $\mathtt{f}_k^{\mathrm{eff}}\big((i - \frac{1}{2})\varepsilon\big)$ on $J_m^L \cup J_m^R$ coincides (up to affine functions) with the flux $\tilde{f}'_k \cup \tilde{f}''_k$ of the two interacting Riemann problems at $(i\varepsilon, m\varepsilon)$, *after the transversal interactions*. Therefore, by definition of the quadratic amount of interactions (see Definition 3.16), we have that

$$\mathtt{A}_k^{\mathrm{quadr}}(i\varepsilon, m\varepsilon) = \bigg\| D \operatorname*{conv}_{J_m} \mathtt{f}_k^{\mathrm{eff}}\Big((i - \frac{1}{2})\varepsilon\Big) - \Big(D \operatorname*{conv}_{J_m^L} \mathtt{f}_k^{\mathrm{eff}}\Big((i - \frac{1}{2})\varepsilon\Big) \cup D \operatorname*{conv}_{J_m^R} \mathtt{f}_k^{\mathrm{eff}}\Big((i - \frac{1}{2})\varepsilon\Big)\Big)\bigg\|_1.$$

Notice now that, by Proposition 3.52, we can apply Proposition 3.65 with $t_1 = t_2 = \big(i - \frac{1}{2}\big)\varepsilon$, $\mathcal{J} = \mathcal{J}_m$, $\mathcal{J}^L = \mathcal{J}_m^L$, $\mathcal{J}^R = \mathcal{J}_m^R$. Therefore

$$\begin{aligned}
\mathtt{A}_k^{\mathrm{quadr}}(i\varepsilon, m\varepsilon) &\le \iint_{\mathcal{J}_m^L \times \mathcal{J}_m^R} \mathtt{q}_k\bigg(\Big(i - \frac{1}{2}\Big)\varepsilon, \Big(i - \frac{1}{2}\Big)\varepsilon, \Big(i - \frac{1}{2}\Big)\varepsilon, w, w'\bigg) dw dw' \\
&\le \iint_{\mathcal{J}_m^L \times \mathcal{J}_m^R} \mathtt{q}_k\bigg(\Big(i - \frac{1}{2}\Big)\varepsilon, \Big(i - \frac{1}{2}\Big)\varepsilon, w, w'\bigg) dw dw' \\
&= \iint_{\mathcal{J}_m^L \times \mathcal{J}_m^R} \mathtt{q}_k\bigg(\Big(i - \frac{1}{2}\Big)\varepsilon, w, w'\bigg) dw dw'.
\end{aligned}$$

Summing over all $m \in \mathbb{Z}$, we get the conclusion. $\square$

CHAPTER 4

Convergence and rate of convergence of the Glimm scheme

In this chapter we prove the second result of this thesis, namely Theorem B in the Introduction. We recall here the statement for the sake of convenience.

Theorem B. *Consider the Cauchy problem*

$$\begin{cases} u_t + F(u)_x = 0 \\ u(t=0) = \bar{u} \end{cases} \tag{4.1}$$

and assume that the system is strictly hyperbolic. Let u^ε be a Glimm approximate solution with mesh size $\varepsilon > 0$ and sampling sequence satisfying

$$\sup_{\lambda \in [0,1]} \left| \lambda - \frac{\mathrm{card}\{i \in \mathbb{N} \mid j_1 \leq i < j_2 \text{ and } \vartheta_i \in [0,\lambda]\}}{j_2 - j_1} \right| \leq C \cdot \frac{1 + \log(j_2 - j_1)}{j_2 - j_1}. \tag{4.2}$$

Denote by $t \mapsto S_t \bar{u}$ the semigroup of vanishing viscosity solutions, provided by Theorem 1 in the Introduction. Then for every fixed time $T \in [0, +\infty)$ the following limit holds:

$$\lim_{\varepsilon \to 0} \frac{\left\| u^\varepsilon(T,\cdot) - S_T \bar{u} \right\|_1}{\sqrt{\varepsilon} |\log \varepsilon|} = 0. \tag{4.3}$$

We widely discussed in the Introduction the history of estimate (4.3) and we also provided there some bibliographical references. Here we just recall that estimate (4.3) was proved by Bressan and Marson in [BM98] in the case of a GNL/LD system. At the best of our knowledge, the proof we present here is the first complete proof of estimate (4.3) in the case of general strictly hyperbolic system, without GNL/LD assumptions. The result of this chapter is contained in paper [MB15].

Structure of the chapter. The chapter is organized as follows.

In Section 4.1 we recall the main points in the proof of estimate (4.3) in the GNL/LD setting, provided by Bressan and Marson in [BM98]. We especially wish to highlight which is the point in Bressan's and Marson's technique which can not be easily extended to the general strictly hyperbolic setting.

In Section 4.2, in the same spirit as [BM98], we construct a wavefront auxiliary map ψ, which will be used to minimize the error in the Glimm scheme due to the restarting procedure on a sufficiently large time interval $[t_1, t_2]$. Our definition of ψ is slightly different from the one in [BM98], because our ψ is constructed backward in time, starting from t_2 and going back towards time t_1. We then construct a wave tracing algorithm for the waves in ψ. Finally we discuss the main properties of ψ, which can be used to prove Theorem B. Such properties are stated in Theorem 4.3, where it is shown that the functional Υ introduced in Chapter 3 bounds the variation in time of the speed of the waves in ψ.

Sections 4.3 and 4.4 are devoted to prove Theorem 4.3. We will follow the same line as in the proof of Theorem A in Chapter 3. In particular Section 4.3 is devoted to prove some local interaction estimate, in the same spirit of the analysis performed in Section 3.2. Section

4.4, on the other hand, is devoted to prove the global part of Theorem 4.3, which is strongly based on the properties of the potential Υ introduced in Chapter 3.

4.1. Bressan's and Marson's technique

We have already recalled in the Introduction (see page XIV) the technique used by Bressan and Marson in [BM98] to prove Theorem B in the GNL/LD case. We wish now to highlight which is the point in Bressan's and Marson's proof which can not be easily extended to the general case, where no assumption of f is made except its strict hyperbolicity, and whose detailed proof is given in this chapter, using the interaction functional Υ introduced in Chapter 3.

Bressan's and Marson's technique is as follows. Thanks to the Lipschitz property of the semigroup S

$$\|S_t\bar{u} - S_s\bar{v}\|_1 \leq L\|\bar{u} - \bar{v}\|_1 + L'|t-s|, \quad \text{for any } \bar{u}, \bar{v} \in \mathcal{D}, \quad t, s \geq 0. \tag{4.4}$$

(see also Theorem 1 in the Introduction), in order to estimate the distance

$$\left\|u^\varepsilon(T,\cdot) - S_T\bar{u}\right\|_{L^1},$$

we can partition the time interval $[0,T]$ in subintervals $J_a := [t_a, t_{a+1}]$ and estimate the error

$$\left\|u^\varepsilon(t_{a+1}) - S_{t_{a+1}-t_a}u^\varepsilon(t_a)\right\|_{L^1} \tag{4.5}$$

on each interval J_a. We have already pointed out in the Introduction that the error (4.5) on J_a comes from two different sources:

(1) first of all there is an error due to the algorithm itself: indeed, in a Glimm approximate solution, roughly speaking, we give each wavefron either speed 0 or speed 1 (according to the sampling sequence $\{\vartheta_i\}_i$), while in the exact solution it would have a speed in $[0,1]$, but not necessarily equal to 0 or 1;
(2) secondly, there is an error due to the fact that some wavefronts can be created at times $t > t_a$, some wavefronts can be canceled at times $t < t_{a+1}$ and, above all, some wavefronts, which are present both at time t_a and at time t_{a+1}, can change their speeds, when they interact with other wavefronts.

The first error source is estimated by choosing the intervals J_a sufficiently large in order to use estimate (2.15) with $j_2 - j_1 \gg 1$.
The second error source can be estimated (choosing the intervals J_a not too large) if we are able to (uniformly) bound the change in speed of the wavefronts present in the approximate solution. In the GNL/LD case, this was achieved by Liu in [Liu77], where he provided a wave tracing algorithm which splits each wavefront in the approximate solution into a finite number of discrete waves, whose trajectories can be traced and whose changes in speed at any interaction time are bounded by the corresponding decrease of the functional Q^{Glimm}. In particular, using Liu's wave tracing, Bressan and Marson prove that for any $i_1, i_2 \in \mathbb{N}$, on the time interval $[t_1, t_2]$, $t_1 = i_1\varepsilon$, $t_2 = i_2\varepsilon$, it holds

$$\left\|u^\varepsilon(t_2) - S_{t_2-t_1}u^\varepsilon(t_1)\right\|_1 \leq \mathcal{O}(1)\left[\left(Q^{\mathrm{Glimm}}(t_2) - Q^{\mathrm{Glimm}}(t_1)\right) + \frac{1+\log(i_2-i_1)}{i_2-i_1} + \varepsilon\right](t_2-t_1). \tag{4.6}$$

Here Q^{Glimm} is the interaction potential introduced by Glimm in [Gli65].
As $\varepsilon \to 0$, it is convenient to choose the asymptotic size of the intervals J_a in such a way that the errors in (1) and (2) above have approximately the same order of magnitude. In particular, the estimate (4.3) is obtained by choosing $|J_a| \approx \sqrt{\varepsilon}\log|\log\varepsilon|$.

Estimate (4.6) is precisely the point in Bressan's and Marson's proof which can not be easily extended to the general case, because the functional Q^{Glimm} is not of help in this case.

In this Chapter we will show that the potential Υ constructed in Chapter 3 has the property that for any $i_1, i_2 \in \mathbb{N}$, $i_1 < i_2$, setting $t_1 := i_1\varepsilon$, $t_2 := i_2\varepsilon$, it holds

$$\left\|u^\varepsilon(t_2) - S_{t_2-t_1}u^\varepsilon(t_1)\right\|_1 \leq \mathcal{O}(1)\left[\Big(\Upsilon(t_2) - \Upsilon(t_1)\Big) + \frac{1+\log(i_2-i_1)}{i_2-i_1}\right](t_2-t_1). \tag{4.7}$$

In order to prove (4.7), one could be tempted to use the well know semigroup inequality (see for instance [Bre00])

$$\|u^\varepsilon(t_2) - S_{t_2-t_1}u^\varepsilon(t_1)\|_1 \leq L\int_{t_1}^{t_2} \limsup_{h\to 0} \frac{\left\|u^\varepsilon(t+h) - S_h u^\varepsilon(t)\right\|_1}{h}dt.$$

However, for a Glimm solution u^ε this estimate can not be directly applied, because it does not take into account the error due to the restarting procedure. To go beyond this difficulty, in the same spirit as in [BM98], we will introduce in Section 4.2 a "wavefront" map

$$\psi : [t_1, t_2] \times \mathbb{R} \to \mathbb{R}^N$$

with the following properties:

$$\psi(t_2, x) = u^\varepsilon(t_2, x), \tag{4.8a}$$

$$\left\|S_{t_2-t_1}\psi(t_1) - \psi(t_2)\right\|_1 \leq \mathcal{O}(1)\left[\Big(\Upsilon(t_1) - \Upsilon(t_2)\Big) + \frac{1+\log(i_2-i_1)}{i_2-i_1}\right](t_2-t_1), \tag{4.8b}$$

$$\left\|\psi(t_1) - u^\varepsilon(t_1)\right\|_1 \leq \mathcal{O}(1)\Big(\Upsilon(t_1) - \Upsilon(t_2)\Big)(t_2-t_1). \tag{4.8c}$$

Clearly (4.7) is an immediate consequence of (4.8) and the Lipschitz continuity of the semigroup S_t.

Remark 4.1. We stress once again that all the functionals $Q^{\mathrm{Glimm}}, Q^{\mathrm{cubic}}, \Upsilon$ are defined on the approximate solution u^ε, or, in other words, they depend on ε, even if we do not write this dependence explicitly. Moreover, the functional Υ introduced in Chapter 3 is defined on an arbitrary finite time $I \subseteq [0, +\infty)$ and not on the whole half line. This is clearly not a problem, since we can choose I sufficiently large in order to contain the interval $[0, T]$. What is important, is that all the functional we are working with are decreasing and uniformly (i.e. without any reference to ε) bounded at $t = 0$.

We conclude this section proving Theorem B in the general case, assuming that estimate (4.7) holds and using Bressan's and Marson's techniques.

Proof of Theorem B, assuming (4.7). Fix $T, \varepsilon > 0$, say $T = \bar{\imath}\varepsilon + \varepsilon'$ for some integer $\bar{\imath}$ and some $\varepsilon' \in [0, \varepsilon)$. In connection with a constant $\delta \geq \varepsilon$ (whose precise value will be specified later), we construct a partition of the interval $[0, \bar{\imath}\varepsilon]$ into finitely many subintervals $J_a = [t_a, t_{a+1}]$, inserting the points $t_a = i_a\varepsilon$ inductively as follows. Set $i_0 := 0$. If the integers $i_0 < i_1 < \cdots < i_a < \bar{\imath}$ have already been defined, then

(i) if $\Upsilon^\varepsilon(i_a\varepsilon) - \Upsilon^\varepsilon\big((i_a+1)\varepsilon\big) \leq \delta$, let i_{a+1} be the largest integer $\leq \bar{\imath}$ such that $(i_{a+1} - i_a)\varepsilon \leq \delta$ and $\Upsilon^\varepsilon(i_a\varepsilon) - \Upsilon^\varepsilon(i_{a+1}\varepsilon) \leq \delta$;

(ii) if $\Upsilon^\varepsilon(i_a\varepsilon) - \Upsilon^a\big((i_a+1)\varepsilon\big) > \delta$, define $i_{a+1} := i_a + 1$.

Clearly $i_A = \bar{\imath}$ for some integer $A \leq \bar{\imath}$. Call $\mathcal{A}', \mathcal{A}''$ respectively the set of indices a for which the alternative (i), (ii) holds. Observe that the cardinalities of these sets can be bounded by

$$\operatorname{card}\mathcal{A}' \leq \mathcal{O}(1)\frac{T}{\delta}, \quad \operatorname{card}\mathcal{A}'' \leq \mathcal{O}(1)\frac{\mathrm{Tot.Var.}(\bar{u})^2}{\delta} \leq \mathcal{O}(1)\frac{T}{\delta} \tag{4.9}$$

for $\delta \ll 1$. On each subinterval J_a, $a \in \mathcal{A}'$ we can apply (4.7), thus obtaining

$$\begin{aligned}&\left\|u^\varepsilon(i_{a+1}\varepsilon) - S_{(i_{a+1}-i_a)\varepsilon}u^\varepsilon(i_a\varepsilon)\right\|_1 \\ &\quad \le \mathcal{O}(1)\left[\Big(\Upsilon^\varepsilon(i_{a+1}\varepsilon) - \Upsilon^\varepsilon(i_a\varepsilon)\Big) + \frac{1+\log(i_{a+1}-i_a)}{i_{a+1}-i_a} + \varepsilon\right](i_{a+1}-i_a)\varepsilon.\end{aligned} \tag{4.10}$$

On the other hand, on each interval J_a with $a \in \mathcal{A}''$, the 1-Lipschitz continuity of $u^\varepsilon : [0,\infty) \to L^1(\mathbb{R};\mathbb{R}^N)$ implies that

$$\left\|u^\varepsilon(i_{a+1}\varepsilon) - S_{(i_{a+1}-i_a)\varepsilon}u^\varepsilon(i_a\varepsilon)\right\|_1 \le (i_{a+1}-i_a)\varepsilon = \varepsilon. \tag{4.11}$$

Using the Lipschitz property (4.4) of the semigroup we get

$$\begin{aligned}&\left\|u^\varepsilon(\bar{i}\varepsilon) - S_{\bar{i}\varepsilon}u^\varepsilon(0)\right\| \\ &\quad \le \sum_{a=0}^{A-1}\left\|S_{(\bar{i}-i_{a+1})\varepsilon}u(i_{a+1}\varepsilon) - S_{(\bar{i}-i_a)\varepsilon}u(i_a\varepsilon)\right\|_1 \\ &\quad \le L\sum_{a=0}^{A-1}\left\|u(i_{a+1}\varepsilon) - S_{(i_{a+1}-i_a)\varepsilon}u(i_a\varepsilon)\right\|_1 \\ &\text{(by (4.10)-(4.11))} \\ &\quad \le \mathcal{O}(1)\left\{\sum_{a\in\mathcal{A}'}\left[\Big(\Upsilon^\varepsilon(i_{a+1}\varepsilon) - \Upsilon^\varepsilon(i_a\varepsilon)\Big) + \frac{1+\log(i_{a+1}-i_a)}{i_{a+1}-i_a} + \varepsilon\right](i_{a+1}-i_a)\varepsilon\right. \\ &\qquad\qquad \left. + \sum_{a\in\mathcal{A}''}\varepsilon\right\} \\ &\text{(by Points (i), (ii) above)} \\ &\quad \le \mathcal{O}(1)\left\{\sum_{a\in\mathcal{A}'}\left(\delta^2 + \varepsilon + \varepsilon\log\frac{\delta}{\varepsilon} + \varepsilon\delta\right) + \sum_{a\in\mathcal{A}''}\varepsilon\right\} \\ \text{(by (4.9))} &\quad \le \mathcal{O}(1)T\left(\delta + \frac{\varepsilon}{\delta} + \frac{\varepsilon}{\delta}\log\frac{\delta}{\varepsilon} + \varepsilon\right)\end{aligned}$$

Hence

$$\begin{aligned}\left\|u^\varepsilon(T) - S_T\bar{u}\right\| &\le \left\|u^\varepsilon(T) - u^\varepsilon(\bar{i}\varepsilon)\right\| + \left\|u^\varepsilon(\bar{i}\varepsilon) - S_{\bar{i}\varepsilon}u^\varepsilon(0)\right\| \\ &\qquad + \left\|S_{\bar{i}\varepsilon}u^\varepsilon(0) - S_{\bar{i}\varepsilon}\bar{u}\right\| + \left\|S_{\bar{i}\varepsilon}\bar{u} - S_T\bar{u}\right\| \\ &\le \mathcal{O}(1)\max\{1,T\}\left(\delta + \frac{\varepsilon}{\delta} + \frac{\varepsilon}{\delta}\log\frac{\delta}{\varepsilon} + \varepsilon\right).\end{aligned} \tag{4.12}$$

Since (4.12) holds for any $\delta \ge \varepsilon$, choosing $\delta(\varepsilon) := \sqrt{\varepsilon}\log|\log\varepsilon|$, we finally obtain (4.3). $\square$

We have just proved Theorem B, assuming (4.7). The remaining part of this chapter is thus devoted to prove that estimate (4.7) actually holds.

4.2. The wavefront map ψ

We have seen in Section 4.1 that a key point to prove Theorem B on the rate of convergence of the Glimm scheme is to construct, for any $i_1, i_2 \in \mathbb{N}$, a map

$$\psi : [i_1\varepsilon, i_2\varepsilon] \times \mathbb{R} \to \mathbb{R}^N$$

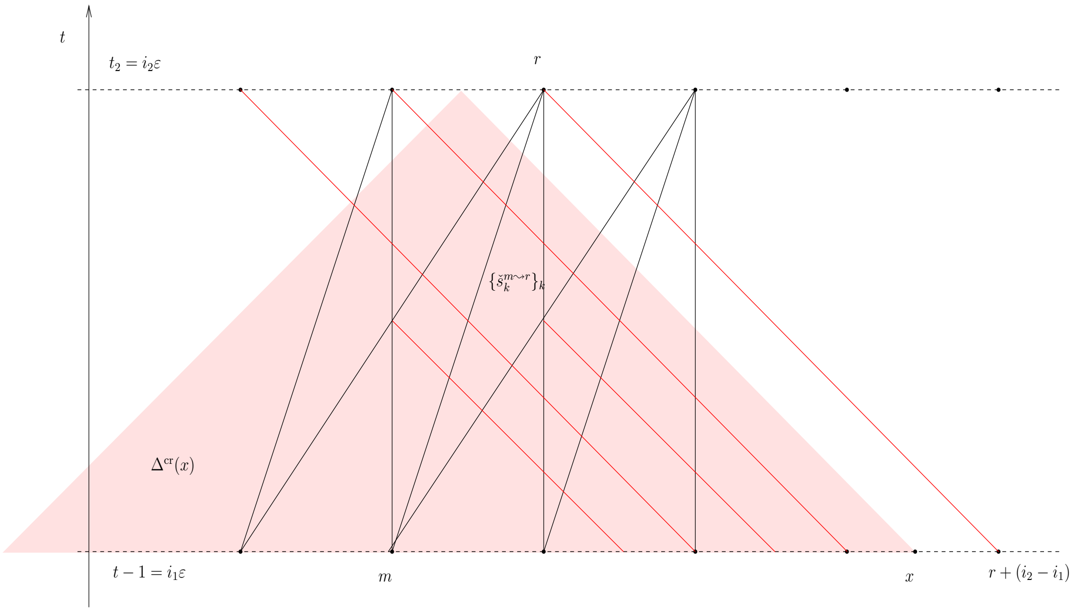

FIGURE 7. The wavefronts of the function ψ: the pink region $\Delta^{\text{cr}}(x)$ is used in the proof of Proposition 4.7.

which satisfies the Properties in (4.8a). In this section we first explicitly define the map ψ, which trivially satisfies (4.8a) and we construct a wave tracing algorithm for the map ψ; then we state the fundamental Theorem 4.3, on the variation in time of the speed of the waves in ψ, whose proof will be the subject of Sections 4.3 and 4.4; finally, using Theorem 4.3, we prove that ψ satisfies also Properties (4.8b) and (4.8c).

4.2.1. Definition of ψ. We start with the explicit definition of ψ, see Figure 7. This map ψ is constructed more or less as in [BM98], with some slight modification. Set for simplicity $t_1 := i_1\varepsilon$ and $t_2 := i_2\varepsilon$. The definition of ψ is given backward in time, starting from time t_2 and going backward to time t_1. First of all we set $\psi(t_2, x) := u^\varepsilon(t_2, x)$ for any $x \in \mathbb{R}$, so that Property (4.8a) is trivially satisfied. Then we define two Riemann solvers, a *starting* RS and a *transversal* RS: both act backward in time and produce a self-similar wavefront solution, with a finite number of wavefronts. The *starting* RS is used at time $t_2 = i_2\varepsilon$ to define ψ on a left neighborhood $[\tilde{t}, t_2]$ of t_2. Then, anytime two wavefronts collide at some time $\bar{t} \in (t_1, t_2)$, assuming that ψ is defined on the time interval $[\bar{t}, t_2]$, we use the *transversal* RS to prolong ψ on a left neighborhood of $\bar{t}$.

The *starting* Riemann Solver. This is the Riemann Solver used at time $t = t_2$. It is defined as follows. For any $m, r \in \mathbb{Z}$, $m \in [r - (i_2 - i_1), r]$, set

$$\check{s}_k^{m\rightsquigarrow r} := \int_{\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2\varepsilon, r\varepsilon)} \rho(t, w)dw, \quad \text{for } t \in [t_1, t_2] \tag{4.13}$$

and the definition is independent of $t \in [t_1, t_2]$ by the regularity properties of ρ in time, Point (4) at page 52. Notice that, by the monotonicity of the map $w \mapsto \mathtt{x}(t, w)$, if $\check{s}_k^{m\rightsquigarrow r}, \check{s}_{k'}^{m\rightsquigarrow r'} \neq 0$ and $r < r'$, then $k \leq k'$. Fix now $r \in \mathbb{Z}$ and for any $m \in [r - (i_2 - i_1), r]$ set

$$\psi^{r-(i_2-i_1)\rightsquigarrow r} := T^N_{\check{s}_N^{r-(i_2-i_1)\rightsquigarrow r}} \circ \cdots \circ T^1_{\check{s}_1^{r-(i_2-i_1)\rightsquigarrow r}} \big(u^{i_2, r-1}\big),$$

$$\psi^{m\rightsquigarrow r} := T^N_{\check{s}_N^{m\rightsquigarrow r}} \circ \cdots \circ T^1_{\check{s}_1^{m\rightsquigarrow r}} \big(\psi^{m-1\rightsquigarrow r}.\big)$$

The (backward) solution to the Riemann problem $(u^{i_2,r-1}, u^{i_2,r})$ is now defined as follows: for any $m = r-(i_2-i_1), \dots, r$ there is a *physical* wavefront traveling with speed

$$\check{\lambda}^{m \rightsquigarrow r} := \frac{r\varepsilon - m\varepsilon}{i_2\varepsilon - i_1\varepsilon} \tag{4.14}$$

which connects the left state $\psi^{m-1 \rightsquigarrow r}$ with the right state $\psi^{m \rightsquigarrow r}$; moreover, there is one more *non-physical* wavefront, traveling with speed equal to $\check{\lambda} := -1$ connecting $\psi^{r \rightsquigarrow r}$ to $u^{i_2,r}$.

The *transversal* Riemann solver. This RS is used every time two (or more) wavefronts collide at a time in (t_1,t_2). We assume w.l.o.g. that every collision involves exactly two wavefronts: the rules can be easily extended to the case of several simultaneous collisions, because the outcome does not depend on the order of the collisions. Assume thus that at point $(\bar{t},\bar{x})$, $\bar{t} \in (t_1,t_2)$ two wavefronts collide. We have to distinguish two cases.

Case 1: both the colliding wavefronts are physical. Assume that before the collision the first wavefront is traveling with speed λ' and it is connecting the states

$$\psi^M = T^N_{s'_N} \circ \dots \circ T^1_{s'_1}\psi^L,$$

while the second wavefront is traveling with speed $\lambda' < \lambda''$ and it is connecting the states

$$\psi^R = T^N_{s''_N} \circ \dots \circ T^1_{s''_1}\psi^M.$$

Notice that, by the monotonicity of the map $w \mapsto \mathtt{x}(t,w)$, there exists $\bar{k} \in \{1,\dots,N\}$ such that $s''_1, \dots, s''_{\bar{k}} = 0$ and $s'_{\bar{k}+1}, \dots, s'_N = 0$. Hence the interaction at $(\bar{t},\bar{x})$ is purely transversal. The (backward) Riemann problem (ψ^L,ψ^R) at point $(\bar{t},\bar{x})$ is now solved as follows. Define the intermediate states

$$\tilde{\psi}^M := T^N_{s''_N} \circ \dots \circ T^1_{s''_{\bar{k}+1}}\psi^L, \qquad \tilde{\psi}^R := T^N_{s'_{\bar{k}}} \circ \dots \circ T^1_{s'_1}\psi^M,$$

The solution for times $t \leq \bar{t}$ around the point $(\bar{t},\bar{x})$ is made by a *physical* wavefront traveling with speed λ'' connecting ψ^L and $\tilde{\psi}^M$; a *physical* wavefront traveling with speed λ' connecting $\tilde{\psi}^M$ and $\tilde{\psi}^R$; a *non-physical* wavefront traveling with speed $\check{\lambda} = -1$ connecting $\tilde{\psi}^R$ and ψ^R.

Case 2: one of the two colliding wavefronts is non-physical. Assume that the non-physical wavefront is connecting ψ^L with ψ^M, while the physical wavefront is traveling with speed λ and it is connecting

$$\psi^R = T^N_{s_N} \circ \dots \circ T^1_{s_1}\psi^M.$$

Define the intermediate state

$$\tilde{\psi}^M := T^N_{s_N} \circ \dots \circ T^1_{s_1}\psi^L.$$

The solution around $(\bar{t},\bar{x})$ for times $t \leq \bar{t}$ is now made by a physical wavefront traveling with speed λ connecting ψ^L with $\tilde{\psi}^M$ and by a non-physical wavefront traveling with speed $\check{\lambda} = -1$ and connecting $\tilde{\psi}^M$ with ψ^R.

It is not difficult to see that the definition of ψ is well posed.

4.2.2. Wave tracing algorithm for ψ. In the same spirit as in Section 3.3 we introduce now a wave tracing algorithm for the wavefront solution ψ. However, contrary to the situation discussed in Section 3.3, we are not interested here in defining a general notion of wave tracing algorithm, since the map ψ is a map *ad hoc* constructed to get estimate (4.6).

First of all, let us analyze the physical waves. For any $k = 1,\dots,N$ the set of the physical waves of the k-th family in ψ is the set $\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$.

Define the position map for the physical waves in ψ as follows:

$$\mathtt{y} : [t_1, t_2] \times \bigcup_{k=1}^{N} \big(\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)\big) \to \mathbb{R}, \quad \mathtt{y}(t, w) := \mathtt{x}(t_2, w) - \frac{\mathtt{x}(t_2, w) - \mathtt{x}(t_1, w)}{t_2 - t_1}(t_2 - t).$$

Notice that $\mathtt{y}$ is defined only for waves in $\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$, i.e. only for waves which have already been created at time t_1 and not yet canceled at time t_2. Moreover, $\mathtt{y}$ takes values in the discontinuity points of ψ, it is increasing in w and affine in t.
We define also the maps $\check{\gamma}(t, \cdot) := (\check{u}(t, \cdot), \check{v}(t, \cdot), \check{\sigma}(t, \cdot))$ at any time $t \in [t_1, t_2]$ as follows. Fix a time t; assume first that no wavefront collision takes place at time t. Fix any wave $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$ and set $x := \mathtt{y}(t, w)$. Assume that

$$u(t, x+) = T^N_{s_N} \circ \cdots \circ T^1_{s_1} u(t, x-);$$

and denote by $\{\gamma_k\}_k$, $\gamma_k = (u_k, v_k, \sigma_k) : \mathbf{I}(s_k) \to \mathbb{R}^{n+2}$ the collection of curves which solve the Riemann problem $(u(t, x-), u(t, x+))$ From the definition of ψ it is not hard to see that

$$s_k = \int_{\mathtt{y}(t)^{-1}(x)} \rho(t, y) dy.$$

Similarly to what we did in (3.30)-(3.31) we can thus define

$$\check{\gamma}(t, \cdot) : \mathtt{y}(t)^{-1}(\bar{x}) \to \mathcal{D}_k \subseteq \mathbb{R}^{n+2}, \qquad \check{\gamma}(t, w) = \big(\check{u}(t, w), \check{v}(t, w), \check{\sigma}(t, w)\big)$$

as

$$\check{\gamma}(t, w) : = \gamma_k\bigg(\int_{\inf((\mathtt{y}(t)^{-1})(x) \cap \mathcal{W}_k(t))}^{w} \rho(t, y) dy\bigg) \text{ for } w \in \mathtt{y}(t)^{-1}(x) \cap \mathcal{W}_k.$$

Notice that we do not need to write an index k in $\check{\gamma}$ (and $\check{u}, \check{v}, \check{\sigma}$), since for any given $w \in \mathcal{W}$ there exists a unique k such that $w \in \mathcal{W}_k$.
Using the fact that, for fixed time t, the position map $\mathtt{y}$ takes values in the discontinuity points of ψ, $\check{\gamma}(t, w)$ is defined for any wave $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$, $k = 1, \dots, N$.
Now, if $t = t_2$ or if t is a time when a collision between two wavefronts takes place, we extend the definitions of $\check{\gamma}(t)$ in order to have left-continuous in time maps, while if $t = t_1$, we extend the definitions of $\check{\gamma}(t)$ by right-continuity.

Remark 4.2. We usually want our maps to be right-continuous in time. In this case, however, we are using backward-in-time Riemann solvers, and thus it is quite natural to require that $t \mapsto \gamma_k(t)$ is left-continuous in time (except at time t_1 when left continuity for the map ψ does not make sense sense).

Finally, we define the *wavefront speed* of a wave $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$ as

$$\check{\lambda}(w) := \frac{\mathtt{x}(i_2\varepsilon, w) - \mathtt{x}(i_1\varepsilon, w)}{i_2\varepsilon - i_1\varepsilon} = \frac{\mathtt{y}(i_2\varepsilon, w) - \mathtt{y}(i_1\varepsilon, w)}{i_2\varepsilon - i_1\varepsilon},$$

which coincides with (4.14).

Let us now analyze the non-physical waves. The set of non-physical wavefront is defined as

$$\mathcal{W}_0 := \big\{(t, x) \;\big|\; \text{in } (t, x) \text{ a non-physical wavefront is generated}\big\}.$$

We are labeling each non-physical wavefront with the point in the (t, x) plane in which it is generated. Since the speed of the non-physical wavefronts is strictly less than the speed of any physical wave, we will refer to the set of non-physical wavefronts also as the set of waves of the 0-th family.

Clearly $\mathcal{W}_0$ is a finite set. For any non-physical wavefronts $\alpha = (\bar{t}, \bar{x}) \in \mathcal{W}_0$, we define its creation time $\mathtt{t}^{\mathrm{cr}}(\alpha) := \bar{t}$ and its position $\mathtt{y}(t, \alpha) = \bar{x} - (t - \bar{t})$. Moreover, if t is any time when no collision between wavefronts takes place, we define the *strength* of the non-physical wavefront α as

$$s(t,\alpha) := \Big|\psi\big(t,\mathtt{y}(t,\alpha)+\big) - \psi\big(t,\mathtt{y}(t,\alpha)-\big)\Big|;$$

then, as usual, we extend the definition to all times in $(t_1,t_2]$ is order to have a left-continuous in time map. Finally define

$$\mathcal{W}_0(t) := \big\{\alpha \in \mathcal{W}_0 \ \big|\ \mathtt{t}^{\mathrm{cr}}(\alpha) \geq t\big\}.$$

We will call $\mathcal{W}_0(t_2)$ the set of *primary* non-physical wavefronts and $\mathcal{W}_0 \setminus \mathcal{W}_0(t_2)$ the set of *secondary* non-physical wavefronts.

4.2.3. The main theorem on ψ. In this section we state the main theorem about physical and non-physical waves in ψ, which will be proved in Sections 4.3 and 4.4, and, using this theorem, we prove estimates (4.8b) and (4.8c).

THEOREM 4.3. *With the same notations as before,*

(1) the following bounds on physical waves hold:

$$\left.\begin{aligned}
&\int_{\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)} \Big\{\mathrm{Tot.Var.}\Big(\check{u}(\cdot,w);(t_1,t_2)\Big) + \Big|\Big(\check{u}(t_2,w) - \bar{u}(t_2,w)\Big)\Big|\Big\}dw \\
&\int_{\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)} \Big\{\mathrm{Tot.Var.}\Big(\check{v}(\cdot,w);(t_1,t_2)\Big) + \Big|\Big(\check{v}(t_2,w) - \bar{v}(t_2,w)\Big)\Big|\Big\}dw \\
&\int_{\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)} \Big\{\mathrm{Tot.Var.}\Big(\check{\sigma}(\cdot,w);(t_1,t_2)\Big) + \Big|\Big(\check{\sigma}(t_2,w) - \bar{\sigma}(t_2,w)\Big)\Big|\Big\}dw
\end{aligned}\right\} \leq \mathcal{O}(1)\Big[\Upsilon(t_1) - \Upsilon(t_2)\Big],$$

where $(\bar{u}(t_2,\cdot),\bar{v}(t_2,\cdot),\bar{\sigma}(t_2,\cdot))$ is the curve solving the exact Riemann problems at time t_2 (i.e. with all waves in $\mathcal{W}_k(i_2\varepsilon, m\varepsilon)$, $m \in \mathbb{Z}$) in the Glimm approximate solution u^ε, see (3.30)-(3.31)*;*

(2) the following bound on non-physical waves holds:

$$\sum_{\alpha\in\mathcal{W}_0} \Big[\mathrm{Tot.Var.}\Big(s(\cdot,\alpha);(t_1,\mathtt{t}^{\mathrm{cr}}(\alpha))\Big) + s\big(\mathtt{t}^{\mathrm{cr}}(\alpha),\alpha\big)\Big] \leq \mathcal{O}(1)\Big[\Upsilon(t_1) - \Upsilon(t_2)\Big].$$

As an immediate consequence, we get the following corollary. For any $k = 1,\dots,N$, for any physical wave $w \in \mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)$ and for any $t \in (t_1,t_2]$, set

$$\check{r}(t,w) := \tilde{r}_k\Big(\check{u}(t,w),\check{v}(t,w),\check{\sigma}(t,w)\Big), \qquad \bar{r}(t,w) := \tilde{r}_k\Big(\bar{u}(t,w),\bar{v}(t,w),\bar{\sigma}(t,w)\Big)$$

As before, also here the index k on $\check{r}$ and $\bar{r}$ is not necessary.

COROLLARY 4.4. *It holds*

$$\int_{\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)} \Big\{\mathrm{Tot.Var.}\Big(\check{r}(\cdot,w);(t_1,t_2)\Big) + \Big|\Big(\check{r}(t_2,w) - \bar{r}_k(t_2,w)\Big)\Big|\Big\}d\tau \leq \mathcal{O}(1)\Big[\Upsilon(t_1) - \Upsilon(t_2)\Big].$$

As we have already said, the proof of Theorem 4.3 is the subject of Sections 4.3 and 4.4. We now use Theorem 4.3 and Corollary 4.4 to prove estimates (4.8b)-(4.8c) and thus complete the proof of Theorem B.

We need first the following lemma, which estimate the distance between the position $\mathtt{x}(t,w)$ of a wave $w \in \mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)$ and the integral of its speed $\sigma(t,w)$ on a time interval $[t_1,t_2]$ in term of the sampling sequence $\{\vartheta_i\}_i$. Recall that $t_1 = i_1\varepsilon$ and $t_2 = i_2\varepsilon$.

LEMMA 4.5. *Let* $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$. *Then*

$$\begin{aligned}
&\left| \mathtt{x}(i_2\varepsilon, w) - \mathtt{x}(i_1\varepsilon, w) - \varepsilon \sum_{i=i_1}^{i_2} \sigma(i\varepsilon, w) \right| \\
&\quad \le 2C(i_2 - i_1)\varepsilon \left[\left(\max_{i=i_1,\dots,i_2-1} \bar\sigma(i\varepsilon, w) - \min_{i=i_1,\dots,i_2-1} \bar\sigma(i\varepsilon, w) \right) + \frac{1 + \log(i_2 - i_1)}{i_2 - i_1} \right].
\end{aligned}$$

PROOF. We use the same technique as in [Liu77]. Define the map $\omega : [0,1] \times [0,1] \to \mathbb{R}$

$$\omega(\sigma, \vartheta) := \begin{cases} -\sigma & \text{if } \sigma \le \vartheta \\ 1 - \sigma & \text{if } \sigma > \vartheta. \end{cases}$$

Using Point (5) in the definition of wave tracing in Section 3.3.1, we can write

$$\begin{aligned}
\mathtt{x}(i_2\varepsilon, w) &- \left(\mathtt{x}(i_1\varepsilon, w) + \varepsilon \sum_{i=i_1}^{i_2-1} \bar\sigma(i\varepsilon, w) \right) \\
&= \varepsilon \sum_{i=i_1}^{i_2-1} \omega\big(\bar\sigma(i\varepsilon, w), \vartheta_i\big) \qquad (4.15) \\
&= \varepsilon \sum_{i=i_1}^{i_2-1} \Big[\omega\big(\bar\sigma(i\varepsilon, w), \vartheta_i\big) - \omega\big(\bar\sigma(i_1\varepsilon, w), \vartheta_i\big) \Big] + \varepsilon \sum_{i=i_1}^{i_2-1} \omega\big(\bar\sigma(i_1\varepsilon, w), \vartheta_i\big).
\end{aligned}$$

Set

$$\sigma^{\min} := \min_{i=i_1,\dots,i_2-1} \bar\sigma(i\varepsilon, w), \qquad \sigma^{\max} := \max_{i=i_1,\dots,i_2-1} \bar\sigma(i\varepsilon, w), \tag{4.16}$$

and

$$\mathcal{J} := \Big\{ i \in [i_1, i_2 - 1] \;\Big|\; \sigma^{\max} \le \vartheta_i \le \sigma^{\min} \Big\}, \qquad \mathcal{K} := \Big\{ i \in [i_1, i_2 - 1] \;\Big|\; \vartheta_i < \sigma(i_1\varepsilon, w) \Big\}.$$

We can continue the computation in (4.15) as follows (here a_i is a number in $\{-1, 0, 1\}$):

$$\begin{aligned}
\dots \;&= \varepsilon \Bigg[\sum_{i \notin \mathcal{J}} \Big(\bar\sigma(i_1\varepsilon, w) - \bar\sigma(i\varepsilon, w) \Big) + \sum_{i \in \mathcal{J}} \Big(\bar\sigma(i_1\varepsilon, w) - \bar\sigma(i\varepsilon, w) + a_i \Big) \\
&\qquad + \sum_{i \notin \mathcal{K}} \Big(- \bar\sigma(i_1\varepsilon, w) \Big) + \sum_{i \in \mathcal{K}} \Big(1 - \bar\sigma(i_1\varepsilon, w) \Big) \Bigg] \\
&= \varepsilon \Bigg[\sum_{i \notin \mathcal{J}} \Big(\bar\sigma(i_1\varepsilon, w) - \bar\sigma(i\varepsilon, w) \Big) + \sum_{i \in \mathcal{J}} \Big(\bar\sigma(i_1\varepsilon, w) - \bar\sigma(i\varepsilon, w) + a_i \Big) \\
&\qquad - \bar\sigma(i_1\varepsilon, w) \Big(i_2 - i_1 - \operatorname{card} \mathcal{K} \Big) + \Big(1 - \bar\sigma(i_1\varepsilon, w) \Big) \operatorname{card} \mathcal{K} \Bigg] \\
&= \varepsilon \Bigg[\sum_{i \notin \mathcal{J}} \Big(\bar\sigma(i_1\varepsilon, w) - \bar\sigma(i\varepsilon, w) \Big) + \sum_{i \in \mathcal{J}} \Big(\bar\sigma(i_1\varepsilon, w) - \bar\sigma(i\varepsilon, w) \pm a_i \Big) \\
&\qquad - \bar\sigma(i_1\varepsilon, w) \Big(i_2 - i_1 \Big) + \operatorname{card} \mathcal{K} \Bigg].
\end{aligned}$$

Hence

$$
\begin{aligned}
&\left| \mathtt{x}(i_2\varepsilon, w) - \left(\mathtt{x}(i_1\varepsilon, w) + \varepsilon \sum_{i=i_1}^{i_2} \bar{\sigma}(i\varepsilon, w) \right) \right| \\
&\quad \le \varepsilon \Bigg[\sum_{i \notin \mathcal{J}} \Big| \bar{\sigma}(i_1\varepsilon, w) - \bar{\sigma}(i\varepsilon, w) \Big| + \sum_{i \in \mathcal{J}} \Big| \bar{\sigma}(i_1\varepsilon, w) - \bar{\sigma}(i\varepsilon, w) + a_i \Big| \\
&\qquad\qquad + \Big| \operatorname{card} \mathcal{K} - \bar{\sigma}(i_1\varepsilon, w) \Big(i_2 - i_1 \Big) \Big| \Bigg] \\
&\quad \le \varepsilon \Bigg[\sum_{i=i_1}^{i_2-1} \Big| \bar{\sigma}(i_1\varepsilon, w) - \bar{\sigma}(i\varepsilon, w) \Big| + \operatorname{card} \mathcal{J} + \Big| \operatorname{card} \mathcal{K} - \bar{\sigma}(i_1\varepsilon, w) \Big(i_2 - i_1 \Big) \Big| \Bigg] \\
&\quad \le \varepsilon \Bigg[\Big(\sigma^{\max} - \sigma^{\min} \Big) \Big(i_2 - i_1 \Big) + \operatorname{card} \mathcal{J} + \Big| \operatorname{card} \mathcal{K} - \bar{\sigma}(i_1\varepsilon, w) \Big(i_2 - i_1 \Big) \Big| \Bigg] \\
&\quad = \varepsilon \Big(i_2 - i_1 \Big) \Bigg[\Big(\sigma^{\max} - \sigma^{\min} \Big) + \frac{\operatorname{card} \mathcal{J}}{i_2 - i_1} + \Bigg| \frac{\operatorname{card} \mathcal{K}}{i_2 - i_1} - \bar{\sigma}(i_1\varepsilon, w) \Bigg| \Bigg] \\
&\quad = \varepsilon \Big(i_2 - i_1 \Big) \Bigg[2 \Big(\sigma^{\max} - \sigma^{\min} \Big) + \Bigg| \frac{\operatorname{card} \mathcal{J}}{i_2 - i_1} - \Big(\sigma^{\max} - \sigma^{\min} \Big) \Bigg| + \Bigg| \frac{\operatorname{card} \mathcal{K}}{i_2 - i_1} - \bar{\sigma}(i_1\varepsilon, w) \Bigg| \Bigg] \\
&\text{(by (2.15))} \\
&\quad \le 2C\varepsilon \Big(i_2 - i_1 \Big) \Bigg[\Big(\sigma^{\max} - \sigma^{\min} \Big) + \frac{1 + \log(i_2 - i_1)}{i_2 - i_1} \Bigg],
\end{aligned}
\tag{4.17}
$$

which concludes the proof of the lemma. □

Let us now prove estimates (4.8b) and (4.8c).

PROPOSITION 4.6 (Estimate (4.8b)). *It holds*

$$
\big\| S_{t_2 - t_1} \psi(t_1) - \psi(t_2) \big\|_1 \le \mathcal{O}(1) \Bigg[\Big(\Upsilon(t_1) - \Upsilon(t_2) \Big) + \frac{1 + \log(i_2 - i_1)}{i_2 - i_1} \Bigg] (t_2 - t_1).
$$

PROOF. We make use the semigroup estimate

$$
\big\| \psi(t_2) - S_{t_2 - t_1} \psi(t_1) \ \big\|_1 \le L \int_{t_1}^{t_2} \limsup_{h \to 0} \frac{\big\| \psi(t+h) - S_h \psi(t) \big\|_1}{h} dt. \tag{4.18}
$$

Since the map ψ is piecewise constant at any fixed time t, it is not hard to see that the integrand on the r.h.s. can be estimated as

$$
\limsup_{h \to 0} \frac{\big\| \psi(t+h) - S_h \psi(t) \big\|_1}{h} \le \sum_{k=1}^{N} \int_{\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)} \big| \check{\lambda}(w) - \check{\sigma}(t, w) \big| dw + 2 \sum_{\alpha \in \mathcal{W}_0(t)} s(t, \alpha).
$$

For the term concerning the non-physical waves, we easily obtain

$$
\begin{aligned}
\sum_{\alpha\in\mathcal{W}_0(t)} s(t,\alpha) &\le \sum_{\alpha\in\mathcal{W}_0(t)} \big|s(t,\alpha)-s\big(\mathtt{t}^{\mathrm{cr}}(\alpha),\alpha\big)\big| + s\big(\mathtt{t}^{\mathrm{cr}}(\alpha),\alpha\big)\\
&\le \sum_{\alpha\in\mathcal{W}_0}\Big[\mathrm{Tot.Var.}\Big(s(\cdot,\alpha);\big(t_1,\mathtt{t}^{\mathrm{cr}}(\alpha)\big)\Big)+s\big(\mathtt{t}^{\mathrm{cr}}(\alpha),\alpha\big)\Big]\\
\text{(by Theorem 4.3)}\quad &\le \mathcal{O}(1)\Big[\Upsilon(t_1)-\Upsilon(t_2)\Big].
\end{aligned}
$$

For the term concerning the physical waves, we argue as follows. Fix any $w\in\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)$. We have

$$
\begin{aligned}
&\big|\check\lambda(w)-\check\sigma(t,w)\big|\\
&\le\bigg|\check\lambda(w)-\frac{1}{i_2-i_1}\sum_{i=i_1}^{i_2-1}\bar\sigma(i\varepsilon,w)\bigg|+\bigg|\frac{1}{i_2-i_1}\sum_{i=i_1}^{i_2-1}\bar\sigma(i\varepsilon,w)-\bar\sigma(i_2\varepsilon,w)\bigg|+\Big|\bar\sigma(i_2\varepsilon,w)-\check\sigma(t,w)\Big|\\
&\le\bigg|\check\lambda(w)-\frac{1}{i_2-i_1}\sum_{i=i_1}^{i_2-1}\bar\sigma(i\varepsilon,w)\bigg|+\mathrm{Tot.Var.}\Big(\bar\sigma(\cdot,w);\Big(t_1,t_2+\frac{\varepsilon}{2}\Big)\Big)\\
&\qquad+\big|\bar\sigma(t_2,w)-\check\sigma(t_2,w)\big|+\mathrm{Tot.Var.}\Big(\check\sigma(\cdot,w);\big(t_1,t_2\big)\Big).
\end{aligned}
\tag{4.19}
$$

The first term can now be estimated using Lemma 4.5, with $\sigma^{\max},\sigma^{\min}$ defined as in (4.16). We thus have

$$
\begin{aligned}
&\bigg|\check\lambda(w)-\frac{1}{i_2-i_1}\sum_{i=i_1}^{i_2-1}\bar\sigma(i\varepsilon,w)\bigg|\\
&\qquad\le\bigg(2C\Big|\sigma^{\max}-\sigma^{\min}\Big|+\frac{1+\log(i_2-i_1)}{i_2-i_1}\bigg)\\
&\qquad\le\mathcal{O}(1)\bigg[\mathrm{Tot.Var.}\Big(\bar\sigma(\cdot,w);\Big(t_1,t_2+\frac{\varepsilon}{2}\Big)\Big)+\frac{1+\log(i_2-i_1)}{i_2-i_1}\bigg].
\end{aligned}
\tag{4.20}
$$

Using (4.19), (4.20), Corollary 3.59 and Theorem 4.3 we thus get

$$
\begin{aligned}
&\int_{\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)}\Big|\check\lambda(w)-\check\sigma(t,w)\Big|d\tau\\
&\qquad\le\mathcal{O}(1)\int_{\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)}\bigg\{\frac{1+\log(i_2-i_1)}{i_2-i_1}+\mathrm{Tot.Var.}\Big(\bar\sigma(\cdot,w);\Big(t_1,t_2+\frac{\varepsilon}{2}\Big)\Big)\\
&\qquad\qquad+\Big|\bar\sigma(t_2,w)-\check\sigma(t_2,w)\Big|+\mathrm{Tot.Var.}\Big(\check\sigma(\cdot,w);\big(t_1,t_2\big)\Big)\bigg\}d\tau\\
&\qquad\le\mathcal{O}(1)\bigg\{\frac{1+\log(i_2-i_1)}{i_2-i_1}+\Upsilon(t_1)-\Upsilon(t_2)\bigg\}
\end{aligned}
$$

Therefore, using (4.18), integrating over all times $t\in[i_1\varepsilon,i_2\varepsilon]$ we get the conclusion. □

PROPOSITION 4.7 (Estimate (4.8c)). *It holds*

$$
\big\|\psi(t_1)-u^\varepsilon(t_1)\big\|_1\le\mathcal{O}(1)\big(\Upsilon(t_1)-\Upsilon(t_2)\big)(t_2-t_1).
$$

Proof. Fix any $x \in \mathbb{R}$. Consider the segment on the (t,x)-plane joining (t_1, x) and $(t_2, x-(t_2-t_1))$. Assume that $x \notin \mathbb{Z}\varepsilon$ and that no non-physical wavefront travels on this segment (this holds for all but finitely many $x \in \mathbb{R}$). Define the set of k-waves which cross this segment in u^ε and in ψ respectively:

$$\begin{aligned} \mathcal{W}_k^{\text{cross}}(u^\varepsilon, x) := \Big\{ w \in \mathcal{W}_k \;\Big|\; & \text{there exists } t =: \mathtt{t}^{\text{cross}}(u^\varepsilon, x, w) \in (t_1, t_2) \\ & \text{such that } \mathtt{x}(t,w) = x - (t - t_1) \Big\} \end{aligned}$$

$$\begin{aligned} \mathcal{W}_k^{\text{cross}}(\psi, x) := \Big\{ w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2) \;\Big|\; & \text{there exists } t =: \mathtt{t}^{\text{cross}}(\psi, x, w) \in (t_1, t_2) \\ & \text{such that } w \in \mathcal{W}_k(t) \text{ and } \mathtt{y}(t,w) = x - (t - t_1) \Big\}. \end{aligned}$$

Since, for any wave $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$, $\mathtt{x}(t_1, w) = \mathtt{y}(t_1, w)$ and $\mathtt{x}(t_2, w) = \mathtt{y}(t_2, w)$,

$$\mathcal{W}_k^{\text{cross}}(\psi, x) = \mathcal{W}_k^{\text{cross}}(u^\varepsilon, x) \cap \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2).$$

Moreover, if a k-wave $w \in \mathcal{W}_k^{\text{cross}}(\psi, x)$, then its position at time t_1 must be

$$\mathtt{x}(t_1, w) = \mathtt{y}(t_1, w) \in \big[x - 2(t_2 - t_1), x\big],$$

while if $w \in \mathcal{W}_k^{\text{cross}}(u^\varepsilon, x) \setminus \mathcal{W}_k^{\text{cross}}(\psi, x)$, then either it is created at some grid point in the triangle

$$\Delta^{\text{cr}}(x) := \Big[\big(t_1, x - 2(t_2 - t_1)\big), \big(t_2, x - (t_2 - t_1)\big), \big(t_1, x\big) \Big]$$

or it is canceled at some grid point in the triangle

$$\Delta^{\text{canc}}(x) := \Big[\big(t_2, x - (t_2 - t_1)\big), \big(t_1, x\big), \big(t_2, x + (t_2 - t_1)\big) \Big]$$

(see Figure 7).

Since $\psi(t_2) = u^\varepsilon(t_2)$, we can now write

$$
\begin{aligned}
&\bigl|\psi(t_1,x) - u^\varepsilon(t_1,x)\bigr| \\
&\qquad = \Bigl|\Bigl[\psi(t_1,x) - \psi\bigl(t_2, x-(t_2-t_1)\bigr)\Bigr] - \Bigl[u^\varepsilon(t_1,x) - u^\varepsilon\bigl(t_2, x-(t_2-t_1)\bigr)\Bigr]\Bigr| \\
&\qquad = \left|\sum_{k=1}^{N} \int_{\mathcal{W}_k^{\text{cross}}(\psi,x)} \Bigl\{\check{r}\Bigl(\mathtt{t}^{\text{cross}}(\psi,x,w),w\Bigr) - \bar{r}\Bigl(\mathtt{t}^{\text{cross}}(u^\varepsilon,x,w),w\Bigr)\Bigr\} dw\right| \\
&\qquad\qquad\qquad + \mathcal{O}(1)\Biggl\{\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\text{cr}}(x)}} \mathtt{A}^{\text{cr}}(i\varepsilon,m\varepsilon) + \sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\text{canc}}(x)}} \mathtt{A}^{\text{canc}}(i\varepsilon,m\varepsilon)\Biggr\} \\
&\qquad \leq \sum_{k=1}^{N} \int_{\mathcal{W}_k^{\text{cross}}(\psi,x)} \Biggl\{\Bigl|\check{r}\Bigl(\mathtt{t}^{\text{cross}}(\psi,x,w),w\Bigr) - \check{r}(t_2,w)\Bigr| \\
&\qquad\qquad\qquad + \Bigl|\check{r}(t_2,w) - \bar{r}(t_2,w)\Bigr| + \Bigl|\bar{r}(t_2,w) - \bar{r}\Bigl(\mathtt{t}^{\text{cross}}(u^\varepsilon,x,w),w\Bigr)\Bigr|\Biggr\} dw \\
&\qquad\qquad + \mathcal{O}(1)\Biggl\{\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\text{cr}}(x)}} \mathtt{A}^{\text{cr}}(i\varepsilon,m\varepsilon) + \sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\text{canc}}(x)}} \mathtt{A}^{\text{canc}}(i\varepsilon,m\varepsilon)\Biggr\} \\
&\qquad \leq \sum_{k=1}^{N} \int_{\mathtt{x}^{-1}([x-2(t_2-t_1),x])} \Biggl\{\Bigl|\text{Tot.Var.}\Bigl(\check{r}(\cdot,w);(t_1,t_2)\Bigr) \\
&\qquad\qquad\qquad + \Bigl|\check{r}(t_2,w) - \bar{r}(t_2,w)\Bigr| + \Bigl|\text{Tot.Var.}\Bigl(\bar{r}(\cdot,w);(t_1,t_2)\Bigr)\Bigr|\Biggr\} dw \\
&\qquad\qquad + \mathcal{O}(1)\Biggl\{\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\text{cr}}(x)}} \mathtt{A}^{\text{cr}}(i\varepsilon,m\varepsilon) + \sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\text{canc}}(x)}} \mathtt{A}^{\text{canc}}(i\varepsilon,m\varepsilon)\Biggr\}.
\end{aligned}
$$

Hence, integrating over all $x \in \mathbb{R}$, we get

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \bigl|\psi(t_1,x) - u^\varepsilon(t_1,x)\bigr| dx \\
&\qquad \leq \int_{-\infty}^{+\infty} \sum_{k=1}^{N} \int_{\mathtt{x}^{-1}([x-2(t_2-t_1),x])} \Biggl\{\Bigl|\text{Tot.Var.}\Bigl(\check{r}(\cdot,w);(t_1,t_2)\Bigr) \\
&\qquad\qquad\qquad + \Bigl|\check{r}(t_2,w) - \bar{r}(t_2,w)\Bigr| + \Bigl|\text{Tot.Var.}\Bigl(\bar{r}(\cdot,w);(t_1,t_2)\Bigr)\Bigr|\Biggr\} dw\, dx \\
&\qquad\quad + \mathcal{O}(1)\int_{-\infty}^{+\infty}\Biggl\{\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\text{cr}}(x)}} \mathtt{A}^{\text{cr}}(i\varepsilon,m\varepsilon) + \sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\text{canc}}(x)}} \mathtt{A}^{\text{canc}}(i\varepsilon,m\varepsilon)\Biggr\} dx
\end{aligned}
$$

(using Fubini's Theorem and Corollaries 3.59 and 4.4)

$$
\leq \mathcal{O}(1)\Bigl[\Upsilon(t_1) - \Upsilon(t_2)\Bigr](t_2 - t_1),
$$

which is what we wanted to get. □

In order to complete the proof of Theorem B, we have thus still to prove Theorem 4.3.

4.3. Analysis of the interactions in ψ

In this and next section we prove Theorem 4.3. We will follow the same technique we used in Chapter 3 to prove Theorem A. In particular this section is devoted to study the *local* part of the theorem: we introduce a suitable notion of amount of interaction and we prove that at any interaction the variation of $\check{u}, \check{v}, \check{\sigma}$ is bounded by such amount of interaction.

In the next section, we will prove the *global* part of the theorem, i.e. that the sum of all the amounts of interactions is bounded by the decrease of Υ in the time interval $[t_1, t_2]$.

4.3.1. Interactions at the final time t_2. We define now some suitable amounts of interaction which bounds the change of $\check{u}, \check{v}, \check{\sigma}$ at time t_2. Fix $r \in \mathbb{R}$. Define first the *transversal amount of interaction* at $(i_2\varepsilon, r\varepsilon)$ as

$$\mathtt{B}^{\text{trans}}(i_2\varepsilon, r\varepsilon) := \sum_{m<m'} \sum_{k>k'} |\check{s}_k^{m \rightsquigarrow r}||\check{s}_{k'}^{m' \rightsquigarrow r}|.$$

Define the *amount of creation* at $(i_2\varepsilon, m\varepsilon)$ as

$$\mathtt{B}^{\text{cr}}(i_2\varepsilon, r\varepsilon) := \mathcal{L}^1\Big(\mathcal{W}_k(i_2\varepsilon, r\varepsilon) \setminus \mathcal{W}_k(t_1)\Big).$$

Now, for any $m \in \mathbb{Z}$, $m \in [r - (i_2 - i_1), r]$ set

$$\mathcal{J}_k^{m \rightsquigarrow r} := \Big\{ w \in \mathcal{W}_k(t_2) \ \Big| \ \mathtt{x}(t_1, w) = m\varepsilon, \ \mathtt{x}(t_2, w) = r\varepsilon \Big\}.$$

Since $\mathtt{x}(t, \cdot)$ is increasing (Property (1) in the definition of wave tracing, Section 3.3.1), $\mathcal{J}_k^{m \rightsquigarrow r}$ is an interval of waves at time t_2 (see Definition 3.24). Recall that $\mathtt{x}(t, w)$ is defined even if $\rho(t, w) = 0$. Therefore, by Proposition 3.32, $J_k^{m \rightsquigarrow r} := \Phi_k(t_2)(\mathcal{J}_k^{m \rightsquigarrow r})$ is an interval in $\mathbb{R}$. Notice also that

$$\mathcal{W}_k(i_2\varepsilon, r\varepsilon) = \bigcup_{m=r-(i_2-i_1)}^{r} \mathcal{J}_k^{m \rightsquigarrow r}$$

and set $J_k^r := \Phi_k(t_2)(\mathcal{W}_k(i_2\varepsilon, r\varepsilon))$. Define the *quadratic amount of interaction* $\mathtt{B}^{\text{quadr}}(i_2\varepsilon, r\varepsilon)$ at $(i_2\varepsilon, r\varepsilon)$ as

$$\mathtt{B}_k^{\text{quadr}}(i_2\varepsilon, r\varepsilon) := \begin{cases} \Big\| D\operatorname{conv}_{J_k^r} \mathtt{f}_k^{\text{eff}}(t_2) - \bigcup_{m=r-(i_2-i_1)}^{r} D\operatorname{conv}_{J_k^{m \rightsquigarrow r}} \mathtt{f}_k^{\text{eff}}(t_2) \Big\|_1 & \text{if } s_k^{i_2, r} \geq 0, \\ \Big\| D\operatorname{conc}_{J_k^r} \mathtt{f}_k^{\text{eff}}(t_2) - \bigcup_{m=r-(i_2-i_1)}^{r} D\operatorname{conc}_{J_k^{m \rightsquigarrow r}} \mathtt{f}_k^{\text{eff}}(t_2) \Big\|_1 & \text{if } s_k^{i_2, r} < 0. \end{cases} \tag{4.21}$$

Proposition 4.8. *It holds*

$$\left.\begin{aligned} &\Big\| \check{u}(t_2-) - \bar{u}(t_2+) \Big\|_{L^1(\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2))} \\ &\Big\| \check{v}(t_2-) - \bar{v}(t_2+) \Big\|_{L^1(\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2))} \\ &\Big\| \check{\sigma}(t_2-) - \bar{\sigma}(t_2+) \Big\|_{L^1(\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2))} \end{aligned}\right\} \leq \mathcal{O}(1) \sum_{r \in \mathbb{Z}} \mathtt{B}(i_2\varepsilon, r\varepsilon),$$

where $\bar{u}(t_2+, w), \bar{v}(t_2+, w), \bar{\sigma}(t_2+, w)$ are defined in (3.30) and (3.31).

Since, at time t_2, $\check{u}$, $\check{v}$, $\check{\sigma}$ are, by definition, left continuous in time, while $\bar{u}$, $\bar{v}$, $\bar{\sigma}$ are right continuous in time, we could have written $\|\check{u}(t_2) - \bar{u}(t_2)\|_1$: the sign $\pm$ are thus just to make the statement clearer.

Proof. It is sufficient to prove that the proposition holds on each $(i_2\varepsilon, r\varepsilon)$, $r \in \mathbb{Z}$, i.e. that for any $r \in \mathbb{R}$

$$\left.\begin{array}{l}\Big\|\check{u}(t_2-) - \bar{u}(t_2+)\Big\|_{L^1(\mathcal{W}_k(t_1)\cap\mathcal{W}_k(i_2\varepsilon,r\varepsilon))}\\ \Big\|\check{v}(t_2-) - \bar{v}(t_2+)\Big\|_{L^1(\mathcal{W}_k(t_1)\cap\mathcal{W}_k(i_2\varepsilon,r\varepsilon))}\\ \Big\|\check{\sigma}(t_2-) - \bar{\sigma}(t_2+)\Big\|_{L^1(\mathcal{W}_k(t_1)\cap\mathcal{W}_k(i_2\varepsilon,r\varepsilon))}\end{array}\right\} \leq \mathcal{O}(1)\mathtt{B}(i_2\varepsilon, r\varepsilon).$$

Fix thus $r \in \mathbb{R}$. Notice that either all the $\check{s}_k^{m\rightsquigarrow r}$ for $m = r-(i_2-i_1), \dots, r$ are greater or equal than zero or they are all less or equal than zero. Assume that they are all ≥ 0. The negative case is completely similar. Define the auxiliary map

$$\Psi_k : \mathcal{W}_k(i_2\varepsilon, r\varepsilon) \cap \mathcal{W}_k(t_1) \to \mathbb{R}, \qquad \Psi(w) := \int_{\inf \mathcal{W}_k(i_2\varepsilon,r\varepsilon)}^{w} \rho(t_1,y)\rho(t_2,y)dy.$$

Using the monotonicity of $\mathtt{x}$, it is not difficult to prove that, for any $m = r-(i_2-i_1), \dots, r$, $\Psi_k(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2, r\varepsilon))$ is an interval in $\mathbb{R}$ and, moreover,

$$\mathcal{L}^1\Big(\Psi_k\big(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2, r\varepsilon)\big)\Big) = \mathcal{L}^1\Big(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2, r\varepsilon)\Big). \tag{4.22}$$

Now let us consider the collection of $N(i_2 - i_1 + 1)$ curves

$$\big\{\gamma_k^{m\rightsquigarrow r} \mid \quad k = 1, \dots, N, \quad m = r-(i_2-i_1), \dots, r\big\}$$

which are interacting at $(i_2\varepsilon, r\varepsilon)$, defined by the following three properties:

(1) the starting point of the first curves $\gamma_1^{r-(i_2-i_1)\rightsquigarrow r}$ is $u^{i_2,r-1}$;
(2) for any m, $\gamma_k^{m\rightsquigarrow r}$ is an exact curve of the k-th family with length $\check{s}_k^{m\rightsquigarrow r}$;
(3) the curves $\{\gamma_k^{m\rightsquigarrow r}\}$, $k = 1, \dots, N$, $m = r-(i_2-i_1), \dots, r$ are consecutive w.r.t. the order

$$(m,k) \text{ precedes } (m',k') \iff m < m' \text{ or } m = m' \text{ and } k < k'.$$

Let us denote the components of $\gamma_k^{m\rightsquigarrow r}$ by $\gamma_k^{m\rightsquigarrow r} = (u_k^{m\rightsquigarrow r}, v_k^{m\rightsquigarrow r}, \sigma_k^{m\rightsquigarrow r})$. Since $\check{s}_k^{m\rightsquigarrow r} \geq 0$ for any k and m, we have that

$$\check{s}_k^{m\rightsquigarrow r} = \mathcal{L}^1\big(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2\varepsilon, r\varepsilon)\big).$$

and thus, by (4.22), we can assume that each curve $\gamma_k^{m\rightsquigarrow r}$ is defined on $\Psi_k(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2, r\varepsilon))$. In this way, by the definitions of $\check{u}, \check{v}, \check{\sigma}$, we have that, for any $w \in \mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2\varepsilon, r\varepsilon)$,

$$\check{u}(t_2,w) = u_k^{m\rightsquigarrow r}\big(\Psi(w)\big), \quad \check{v}(t_2,w) = v_k^{m\rightsquigarrow r}\big(\Psi(w)\big), \quad \check{\sigma}(t_2,w) = \sigma_k^{m\rightsquigarrow r}\big(\Psi(w)\big).$$

Similarly, the Riemann problem $(u^{i_2,r-1}, u^{i_2,r})$ in u^ε at $(i_2\varepsilon, r\varepsilon)$ is solved by the curves $\{\gamma_k^{i_2,r}\}$, with $\gamma_k^{i_2,r}$ defined on $s_k^{i_2,r}$. Since

$$s_k^{i_2,r} = \mathcal{L}^1\Big(\mathcal{W}_k(i_2\varepsilon, r\varepsilon)\Big) = \mathcal{L}^1\Big(\Phi_k(t_2)(\mathcal{W}_k(i_2\varepsilon, r\varepsilon)\Big),$$

we can assume that $\gamma_k^{i_2,r}$ is defined on the interval $\Phi_k(t_2)(\mathcal{W}_k(i_2\varepsilon, r\varepsilon))$ and thus, by definition of $\bar{u}(t_2,w), \bar{v}(t_2,w), \bar{\sigma}(t_2,w)$, it holds, for any $k = 1, \dots, N$ and for any $w \in \mathcal{W}_k(i_2\varepsilon, r\varepsilon)$,

$$\bar{u}(t_2,w) = u_k^{i_2,r}\big(\Phi_k(t_2)(w)\big),$$
$$\bar{v}(t_2,w) = u_k^{i_2,r}\big(\Phi_k(t_2)(w)\big),$$
$$\bar{\sigma}(t_2,w) = u_k^{i_2,r}\big(\Phi_k(t_2)(w)\big).$$

Define now the map

$$\Theta_k : \Psi_k\big(\mathcal{W}_k(i_2\varepsilon, r\varepsilon) \cap \mathcal{W}_k(t_1)\big) \to \Phi_k\big(\mathcal{W}_k(i_2\varepsilon, r\varepsilon)\big), \quad \Theta_k := \Phi_k(t_2) \circ \Psi_k^{-1}.$$

It is not hard to see that Θ_k is a piecewise affine map with slope equal to 1. The proof is concluded if we prove the next Lemma. ☐

LEMMA 4.9. *For any $k = 1, \dots, N$, the following inequalities hold*

$$\left.\begin{array}{r}
\left\| \bigcup_{m=r-(i_2-i_1)}^{r} u_k^{m\rightsquigarrow r} - u_k^{i_2,r} \circ \Theta_k \right\|_{L^1(\Psi_k(\mathcal{W}_k(i_2\varepsilon,r\varepsilon)\cap\mathcal{W}_k(t_1)))} \\
\left\| \bigcup_{m=r-(i_2-i_1)}^{r} v_k^{m\rightsquigarrow r} - v_k^{i_2,r} \circ \Theta_k \right\|_{L^1(\Psi_k(\mathcal{W}_k(i_2\varepsilon,r\varepsilon)\cap\mathcal{W}_k(t_1)))} \\
\left\| \bigcup_{m=r-(i_2-i_1)}^{r} \sigma_k^{m\rightsquigarrow r} - \sigma_k^{i_2,r} \circ \Theta_k \right\|_{L^1(\Psi_k(\mathcal{W}_k(i_2\varepsilon,r\varepsilon)\cap\mathcal{W}_k(t_1)))}
\end{array}\right\} \leq \mathcal{O}(1)\mathtt{B}(i_2\varepsilon, r\varepsilon).$$

PROOF. Set, for simplicity,

$$I_k^{m\rightsquigarrow r} := \Psi_k\Big(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2\varepsilon, r\varepsilon)\Big). \tag{4.23}$$

Following the same technique as in Theorem 3.18, the proof is achieved in three steps, using the basic estimates of Section 3.1. We just sketch the proof of each step.

(1) First we perform all the transversal interactions, passing from the collection of $N(i_2-i_1+1)$ curves $\{\gamma_k^{m\rightsquigarrow r}\}$ to the collection of $N(i_2-i_1+1)$ curves $\{\tilde\gamma_k^{m\rightsquigarrow r}\}$, $k = 1, \dots, N$, $m = r-(i_2-i_1), \dots, r$, such that
   (a) the starting point of the first curves $\tilde\gamma_1^{r-(i_2-i_1)\rightsquigarrow r} \rightsquigarrow r$ is $u^{i_2,r-1}$;
   (b) for any m, $\tilde\gamma_k^{m\rightsquigarrow r}$ is an exact curve of the k-th family with length $\check s_k^{m\rightsquigarrow r}$;
   (c) the curves $\{\tilde\gamma_k^{m\rightsquigarrow r}\}$ are consecutive w.r.t. the order

$$(m,k) \text{ precedes } (m',k') \iff k < k' \text{ or } k = k' \text{ and } m < m'. \tag{4.24}$$

As usual, we denote by $\tilde\gamma_k^{m\rightsquigarrow r} = (\tilde u_k^{m\rightsquigarrow r}, \tilde v_k^{m\rightsquigarrow r}, \tilde\sigma_k^{m\rightsquigarrow r})$ the components of $\tilde\gamma_k^{m\rightsquigarrow r}$. We assume that, for fixed k, the collection of curves $\{\gamma_k^{r-(i_2-i_1+1)\rightsquigarrow r}, \dots, \gamma_k^{m\rightsquigarrow r}\}$ satisfies the assumption ($\star$). Using Corollary 3.13, we get

$$\left.\begin{array}{r}
\sum_{k=1}^{N}\sum_{p=1}^{P} \|u_k^{m\rightsquigarrow r} - \tilde u_k^{m\rightsquigarrow r}\|_{L^1(I_k^{m\rightsquigarrow r})} \\
\sum_{k=1}^{N}\sum_{p=1}^{P} \|v_k^{m\rightsquigarrow r} - \tilde v_k^{m\rightsquigarrow r}\|_{L^1(I_k^{m\rightsquigarrow r})} \\
\sum_{k=1}^{N}\sum_{p=1}^{P} \|\sigma_k^{m\rightsquigarrow r} - \tilde\sigma_k^{m\rightsquigarrow r}\|_{L^1(I_k^{m\rightsquigarrow r})}
\end{array}\right\} \leq \mathcal{O}(1) \sum_{p<p'}\sum_{k>k'} \big|s_k^{m\rightsquigarrow r}\big|\big|s_{k'}^{m'\rightsquigarrow r}\big| = \mathcal{O}(1)\mathtt{B}^{\mathrm{trans}}(i_2\varepsilon, r\varepsilon).$$

(2) Then we modify the length of each $\tilde\gamma_k^{m\rightsquigarrow r}$ from $\check s_k^{m\rightsquigarrow r} = \mathcal{L}^1(I_k^{m\rightsquigarrow r})$ to $\mathcal{L}^1(J_k^{m\rightsquigarrow r})$, thus getting a new collection of $N(i_2-i_1+1)$ consecutive (w.r.t. the order in (4.24)) curves $\{\hat\gamma_k^{m\rightsquigarrow r}\}$ such that the starting point of $\hat\gamma_1^{m-(i_2-i_1)\rightsquigarrow r}$ is $u^{i_2,r-1}$ and the length

of $\hat{\gamma}_k^{m\rightsquigarrow r}$ is $\mathcal{L}^1(J_k^{m\rightsquigarrow r})$. Again, the components are $\hat{\gamma}_k^{m\rightsquigarrow r} = (\hat{u}_k^{m\rightsquigarrow r}, \hat{v}_k^{m\rightsquigarrow r}, \hat{\sigma}_k^{m\rightsquigarrow r})$. By Lemmas 3.11 and 3.8, we get

$$\left.\begin{array}{l} \displaystyle\sum_{k,m} \big\|\tilde{u}_k^{m\rightsquigarrow r} - \hat{u}_k^{m\rightsquigarrow r} \circ \Theta_k|_{I_k^{m\rightsquigarrow r}}\big\|_{L^1(I_k^{m\rightsquigarrow r})} \\[2em] \displaystyle\sum_{k,m} \big\|\tilde{v}_k^{m\rightsquigarrow r} - \hat{v}_k^{m\rightsquigarrow r} \circ \Theta_k|_{I_k^{m\rightsquigarrow r}}\big\|_{L^1(I_k^{m\rightsquigarrow r})} \\[2em] \displaystyle\sum_{k,m} \big\|\tilde{\sigma}_k^{m\rightsquigarrow r} - \hat{\sigma}_k^{m\rightsquigarrow r} \circ \Theta_k|_{I_k^{m\rightsquigarrow r}}\big\|_{L^1(I_k^{m\rightsquigarrow r})} \end{array}\right\} \leq \mathcal{O}(1) \sum_{k,p} \mathcal{L}^1\big(J_k^{m\rightsquigarrow r} \setminus I_k^{m\rightsquigarrow r}\big) = \mathcal{O}(1) \sum_{k=1}^N \mathtt{B}_k^{\mathrm{cr}}(i_2\varepsilon, r\varepsilon).$$

(3) Finally we perform all the non transversal interactions passing from the collection of curves $\{\hat{\gamma}_k^{m\rightsquigarrow r}\}$ to $\{\gamma_k\}$. By the second part of Lemma 3.14 and using the fact that the reduced flux $f_k^{i_2,r}$ associated to the curve $\gamma_k^{i_2,r}$ coincides, up to affine functions, with $\mathtt{f}_k^{\mathrm{eff}}(t_2)$, we have

$$\begin{aligned} &\sum_{k,m} \big\|\hat{u}_k^{m\rightsquigarrow r} - u_k\big\|_{L^1(J_k^{m\rightsquigarrow r})}, \quad \sum_{k,m} \big\|\hat{v}_k^{m\rightsquigarrow r} - v_k\big\|_{L^1(J_k^{m\rightsquigarrow r})}, \quad \sum_{k,m} \big\|\hat{\sigma}_k^{m\rightsquigarrow r} - \sigma_k\big\|_{L^1(J_k^{m\rightsquigarrow r})} \\ &\quad \leq \mathcal{O}(1) \bigg\| \frac{d}{d\tau} \operatorname*{conv}_{\mathbf{I}(s_k)} f_k - \bigcup_{a=0}^P \frac{d}{d\tau} \operatorname*{conv}_{J_k^{m\rightsquigarrow r}} f_k \bigg\|_1 \\ &\quad \leq \mathcal{O}(1) \bigg\| \frac{d}{d\tau} \operatorname*{conv}_{J_k^r} \mathtt{f}_k^{\mathrm{eff}}(t_2) - \bigcup_{a=0}^P \frac{d}{d\tau} \operatorname*{conv}_{J_k^{m\rightsquigarrow r}} \mathtt{f}_k^{\mathrm{eff}}(t_2) \bigg\|_1 \\ &\quad = \mathcal{O}(1) \mathtt{B}_k^{\mathrm{quadr}}(i_2\varepsilon, r\varepsilon), \end{aligned}$$

thus concluding the proof of the lemma and, therefore, also the proof of Proposition 4.8. □

4.3.2. Amounts of interaction at times $t \in (t_1, t_2)$. Let $t \in (t_1, t_2)$ and let (t, x) be a point where two wavefronts collide. As in Section 4.2.1, we have to distinguish to two cases.

Case 1: both the colliding wavefronts are physical. Assume that before the collision the first wavefront is traveling with speed λ' and it is connecting the states

$$\psi^M = T^N_{s'_{\bar{k}}} \circ \cdots \circ T^1_{s'_1} \psi^L,$$

while the second wavefront is traveling with speed $\lambda' < \lambda''$ and it is connecting the states

$$\psi^R = T^N_{s''_N} \circ \cdots \circ T^1_{s''_{\bar{k}}} \psi^M.$$

We have already observed that the interaction at $(\bar{t}, \bar{x})$ is purely transversal, i.e. there exists $\bar{k} \in \{1, \dots, N\}$ such that $s''_1, \dots, s''_{\bar{k}} = 0$ and $s'_{\bar{k}+1}, \dots, s'_N = 0$. Define thus the *(transversal) amount of interaction* at (t, x) as

$$\mathtt{B}^{\mathrm{trans}}(t, x) := \sum_{k=1}^{\bar{k}} \sum_{h=\bar{k}+1}^N |s'_k||s''_h|.$$

Case 2: one of the two colliding wavefronts is non-physical. Assume that the non-physical wavefront α is connecting ψ^L with ψ^M, while the physical wavefront is traveling with speed

λ and it is connecting

$$\psi^R = T^N_{s_N} \circ \cdots \circ T^1_{s_1} \psi^M.$$

Also in this case the interaction is purely transversal. Define thus the *amount of interaction* at (t, x) as

$$\mathtt{B}(t, x) := \mathtt{B}^{\mathrm{trans}}(t, x) := s(t+, \alpha) \sum_{k=1}^{N} |s_k| = |\psi^M - \psi^L| \sum_{k=1}^{N} |s_k|.$$

The following proposition covers both the case of a collision between physical wavefronts and the case of a collision between a physical and a non-physical wavefront.

Proposition 4.10. *The following hold.*

(1) For any $k = 1, \dots, N$, for the k-physical waves $\mathtt{y}(t)^{-1}(x) \cap \mathcal{W}_k$ located at (t, x) in the wavefront map ψ, we have

$$\left.\begin{array}{l} \Big\|\check{u}(t+) - \check{u}(t-)\Big\|_{L^1(\mathtt{y}(t)^{-1}(x)\cap\mathcal{W}_k)} \\ \Big\|\check{v}(t+) - \check{v}(t-)\Big\|_{L^1(\mathtt{y}(t)^{-1}(x)\cap\mathcal{W}_k)} \\ \Big\|\check{\sigma}(t+) - \check{\sigma}(t-)\Big\|_{L^1(\mathtt{y}(t)^{-1}(x)\cap\mathcal{W}_k)} \end{array}\right\} \leq \mathcal{O}(1)\mathtt{B}^{\mathrm{trans}}(t, x).$$

(2) If both wavefronts interacting at (t, x) are physical, denoting by α the non-physical wavefront generated at (t, x), its initial strength can be estimated by

$$\Big|s\big(\mathtt{t}^{\mathrm{cr}}(\alpha), \alpha\big)\Big| \leq \mathcal{O}(1)\mathtt{B}^{\mathrm{trans}}(t, x).$$

(3) If one of the two wavefronts interacting at (t, x) is a non-physical wavefront α, the variation of the strength of α can be estimated by

$$|s(t+, \alpha) - s(t-, \alpha)| \leq \mathcal{O}(1)\mathtt{B}^{\mathrm{trans}}(t, x).$$

Proof. If both colliding wavefronts are physical, the conclusion is an immediate consequence of Lemma 3.12. If one of the two wavefronts is non-physical, then the conclusion follows from Lemma 3.7. □

4.4. Estimates on the amounts of interaction in ψ

In this section we prove the following theorem, which is the *global* part of the proof of Theorem 4.3. The proof of this theorem is the last step in order to complete the proof of the convergence rate of the Glimm scheme stated in Theorem B.

Theorem 4.11. *The sum of all amounts of interaction in the time interval $(t_1, t_2]$ is bounded by the decrease of the functional Υ in the same time interval, i.e.*

$$\sum_{r\in\mathbb{Z}} \mathtt{B}(i_2\varepsilon, r\varepsilon) + \sum_{\substack{(t,x) \text{ int. pt.} \\ t\in(t_1,t_2)}} \mathtt{B}^{\mathrm{trans}}(t, x) \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big).$$

The proof is a direct consequence of the following three propositions.

Proposition 4.12 (Transversal amounts of interactions). *It holds*

$$\sum_{r\in\mathbb{Z}} \mathtt{B}^{\mathrm{trans}}(i_2\varepsilon, r\varepsilon) + \sum_{\substack{(t,x) \text{ int. pt.} \\ t\in(t_1,t_2)}} \mathtt{B}^{\mathrm{trans}}(t, x) \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big).$$

PROOF. Since for any wave $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$,

$$\mathtt{x}(t_1, w) = \mathtt{y}(t_1, w), \qquad \mathtt{x}(t_2, w) = \mathtt{y}(t_2, w),$$

and thus the waves which have to cross in ψ also cross in u^ε, it is not difficult to see that

$$\begin{aligned}\sum_{r \in \mathbb{Z}} \mathtt{B}^{\text{trans}}(i_2\varepsilon, r\varepsilon) + \sum_{\substack{(t,x) \text{ int. pt.} \\ t \in (t_1, t_2)}} \mathtt{B}^{\text{trans}}(t, x) &\leq \sum_{i=i_1+1}^{i_2} \sum_{m \in \mathbb{Z}} \mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) \\ (\text{by } (3.51)) &\leq \mathcal{O}(1)\big(\Upsilon(i_2\varepsilon) - \Upsilon(t_1)\big),\end{aligned}$$

which is what we wanted to prove. □

PROPOSITION 4.13 (Amounts of creation). *It holds*

$$\sum_{r \in \mathbb{Z}} \mathtt{B}_k^{\text{cr}}(i_2\varepsilon, r\varepsilon) \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big).$$

PROOF. It is fairly easy to see that

$$\sum_{r \in \mathbb{Z}} \mathtt{B}_k^{\text{cr}}(i_2\varepsilon, r\varepsilon) \leq \sum_{i=i_1+1}^{i_2} \sum_{m \in \mathbb{Z}} \mathtt{A}^{\text{cr}}(i_2\varepsilon, m\varepsilon),$$

and thus, again using (3.51), we get the conclusion. □

PROPOSITION 4.14 (Quadratic amounts of interaction). *It holds*

$$\sum_{r \in \mathbb{Z}} \mathtt{B}_k^{\text{quadr}}(i_2\varepsilon, r\varepsilon) \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big). \tag{4.25}$$

The proof of this proposition requires more work than the previous two.

PROOF. For $m, m', r \in \mathbb{Z}$, $m < m'$, set

$$\begin{aligned}\mathcal{E}_{m,m',r} &:= \big\{(w, w') \;\big|\; w, w' \in \mathcal{W}_k(i_2\varepsilon, r\varepsilon),\ \mathtt{x}(t_1, w) = m\varepsilon,\ \mathtt{x}(t_1, w') = m'\varepsilon\big\} \\ \mathcal{B}_{m,m',r} &:= \big\{(w, w') \in \mathcal{E}_{m,m',r} \;\big|\; w, w' \in \mathcal{W}_k(t_1)\big\} \\ \mathcal{C}_{m,m',r} &:= \mathcal{E}_{m,m',r} \setminus \mathcal{B}_{m,m',r}.\end{aligned}$$

Set also

$$\mathcal{E}_r := \bigcup_{m<m'} \mathcal{E}_{m,m',r}, \quad \mathcal{B}_r := \bigcup_{m<m'} \mathcal{B}_{m,m',r}, \quad \mathcal{C}_r := \bigcup_{m<m'} \mathcal{C}_{m,m',r}$$

and

$$\mathcal{E} := \bigcup_{r \in \mathbb{Z}} \mathcal{E}_r, \quad \mathcal{B} := \bigcup_{r \in \mathbb{Z}} \mathcal{B}_r, \quad \mathcal{C} := \bigcup_{r \in \mathbb{Z}} \mathcal{C}_r.$$

Finally define also

$$\mathcal{B}^i := \Big\{(w, w') \in \mathcal{B} \;\Big|\; \mathtt{t}^{\text{int}}(t_1, w, w') = i\varepsilon\Big\}, \quad i = i_1 + 1, \dots, i_2.$$

The proof is now divided in four steps.
Step 1. For any $r \in \mathbb{Z}$,

$$\mathtt{B}_k^{\text{quadr}}(i_2\varepsilon, r\varepsilon) \leq \mathcal{O}(1) \iint_{\mathcal{E}_r} \mathfrak{q}_k(t_1, t_2, t_2, w, w') dw dw'.$$

Proof of Step 1. We assume for the sake of simplicity that the k-waves interacting at $(i_2\varepsilon, r\varepsilon)$ are positive, the negative case being completely similar. By definition (see (4.21))

$$\mathtt{B}_k^{\mathrm{quadr}}(i_2\varepsilon, r\varepsilon) := \left\| D \operatorname*{conv}_{J_k^r} \mathtt{f}_k^{\mathrm{eff}}(t_2) - \bigcup_{m=r-(i_2-i_1)}^{r} D \operatorname*{conv}_{J_k^{m\rightsquigarrow r}} \mathtt{f}_k^{\mathrm{eff}}(t_2) \right\|_1.$$

By triangular inequality, it is enough to prove that for any $\bar m = r-(i_2-i_1)+1,\dots,r$,

$$\left\| D \operatorname*{conv}_{\bigcup_{m=r-(i_2-i_1)}^{\bar m} J_k^{m\rightsquigarrow r}} \mathtt{f}_k^{\mathrm{eff}}(t_2) - \left(D \operatorname*{conv}_{\bigcup_{m=r-(i_2-i_1)}^{\bar m-1} J_k^{m\rightsquigarrow r}} \mathtt{f}_k^{\mathrm{eff}}(t_2) \cup D \operatorname*{conv}_{J_k^{\bar m\rightsquigarrow r}} \mathtt{f}_k^{\mathrm{eff}}(t_2) \right) \right\|_1$$
$$\leq \iint_{(\bigcup_{m=r-(i_2-i_1)}^{\bar m-1} \mathcal{J}_k^{\bar m-1\rightsquigarrow r}) \times \mathcal{J}_k^{\bar m\rightsquigarrow r}} \mathtt{q}_k\Big(t_1, t_2, t_2, w, w'\Big) dw dw'. \tag{4.26}$$

Now observe that inequality (4.26) is a consequence of Proposition 3.65 if, for any $m = r-(i_2-i_1),\dots,\bar m-1$ and $(w,w') \in \mathcal{J}_k^{m\rightsquigarrow r} \times \mathcal{J}_k^{\bar m\rightsquigarrow r}$, the following two conditions are satisfied:

a) $\mathtt{p}(t_1, w, w')$ holds;
b) $\mathcal{P}(t_1, t_2, w, w')$ can be restricted both to $\mathcal{J}_k^{m\rightsquigarrow r}$ and to $\mathcal{J}_k^{\bar m\rightsquigarrow r}$.

Condition a) is an immediate consequence of the definition of the sets $\mathcal{J}_k^{m\rightsquigarrow r}$. To prove condition b) argue as follows. If at least one between w, w' does not belong to $\mathcal{W}_k(t_1)$, then each element of $\mathcal{P}(t_1, t_2, w, w')$ is a singleton and thus the proof is trivial. Assume now that $w, w' \in \mathcal{W}_k(t_1)$. Take $\mathcal{K} \in \mathcal{P}(t_1, t_2, w, w')$ with $\mathcal{K} \cap \mathcal{J}_k^{m\rightsquigarrow r} \neq \emptyset$, $z \in \mathcal{K} \cap \mathcal{J}_k^{m\rightsquigarrow r}$. We want to prove that $\mathcal{K} \subseteq \mathcal{J}_k^{m\rightsquigarrow r}$. If $z \notin \mathcal{W}_k(t_1)$, then $\mathcal{K} = \{z\}$ and thus we are done. On the other side, if $z \in \mathcal{W}_k(t_1)$, pick $z' \in \mathcal{K}$. Proving that $z' \in \mathcal{J}_k^{m\rightsquigarrow r}$ means proving that $\mathtt{x}(t_1, z') = m\varepsilon$ and $\mathtt{x}(t_2, z') = r\varepsilon$. Since z, z' belong to the same equivalence class in $\mathcal{P}(t_1, t_2, w, w')$, then by Proposition 3.52 they must have the same position at time t_2 and thus $\mathtt{x}(t_2, z') = r\varepsilon$. Now notice that, by definition of the partition, z, z' must belong to the same equivalence class also in the partition $\mathcal{P}(t_1, t_1, w, w')$ and thus, again by Proposition 3.52, they must have the same position also at time t_1, i.e. $\mathtt{x}(t_1, z') = m\varepsilon$, thus proving that $z' \in \mathcal{J}_k^{m\rightsquigarrow r}$ and $\mathcal{K} \subseteq \mathcal{J}_k^{m\rightsquigarrow r}$. Since both a) and b) hold, we can apply Proposition 3.65 to get (4.26).

Step 2. The integral over pair of waves such that at least one of the two waves is created after time t_1 is estimated by:

$$\iint_{\mathcal{C}} \mathtt{q}_k(t_1, t_2, t_2, w, w') dw dw' \leq \mathcal{O}(1) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}_k^{\mathrm{cr}}(i\varepsilon, m\varepsilon).$$

Proof of Step 2 The proof is an easy consequence of the definition of the set $\mathcal{C}$ and the fact that the weights $\mathtt{q}_k$ are uniformly bounded, Remark 3.56.

Step 3. It holds

$$\iint_{\mathcal{B}} \left[\mathtt{q}_k(t_1, t_2, t_2, w, w') - \mathtt{q}_k\Big(\mathtt{t}^{\mathrm{int}}(t_1, w, w') - \varepsilon, w, w'\Big) \right] dw dw' \leq \mathcal{O}(1) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon).$$

Proof of Step 3. Define for any pair of waves $w, w' \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$ having the same sign the quantity

$$\Psi(w,w') := \begin{cases} \displaystyle\int_w^{w'} \big[\rho(t_1,y)\rho(t_2,y)\big]^+ dy & \text{if } \mathcal{S}(w) = \mathcal{S}(w') = 1, \\[2ex] \displaystyle\int_w^{w'} -\big[\rho(t_1,y)\rho(t_2,y)\big]^- dy & \text{if } \mathcal{S}(w) = \mathcal{S}(w') = -1, \end{cases}$$

describing the mass of waves in $\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$ which stay between w and w'. Observe that for any $j = i_1, \dots, i_2$,

$$\Big|\Phi_k(j\varepsilon)(w') - \Phi_k(j\varepsilon)(w)\Big| \geq \Psi_k(w,w'). \tag{4.27}$$

Now notice that

$$\begin{aligned} \mathtt{q}\Big(\mathtt{t}^{\text{int}}(t_1,w,w') - \varepsilon, w, w'\Big) &= \mathtt{q}\Big(\mathtt{t}^{\text{int}}(t_1,w,w') - \varepsilon, \mathtt{t}^{\text{int}}(t_1,w,w') - \varepsilon, w, w'\Big) \\ &= \mathtt{q}\Big(t_1, \mathtt{t}^{\text{int}}(t_1,w,w') - \varepsilon, w, w'\Big) \\ &\geq \mathtt{q}\Big(t_1, \mathtt{t}^{\text{int}}(t_1,w,w') - \varepsilon, t_2, w, w'\Big). \end{aligned}$$

Hence

$$\begin{aligned} \Delta\mathtt{q}_k(w,w') &= \mathtt{q}\Big(t_1,t_2,t_2,w,w'\Big) - \mathtt{q}\Big(\mathtt{t}^{\text{int}}(t_1,w,w') - \varepsilon, w, w'\Big) \\ &\leq \mathtt{q}\Big(t_1,t_2,t_2,w,w'\Big) - \mathtt{q}\Big(t_1, \mathtt{t}^{\text{int}}(t_1,w,w') - \varepsilon, t_2, w, w'\Big) \\ &\leq \sum_{i=\mathtt{t}^{\text{int}}(t_1,w,w')/\varepsilon}^{i_2} \Big[\mathtt{q}\Big(t_1, i\varepsilon, t_2, w, w'\Big) - \mathtt{q}\Big(t_1, (i-1)\varepsilon, t_2, w, w'\Big)\Big] \\ &\qquad \text{(by Lemma 3.63)} \\ &\leq \mathcal{O}(1) \sum_{i=\mathtt{t}^{\text{int}}(t_1,w,w')/\varepsilon}^{i_2} \frac{1}{|\Phi_k((i-1)\varepsilon)(w') - \Phi_k((i-1)\varepsilon)(w)|} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \\ &\qquad \text{(by (4.27))} \\ &\leq \mathcal{O}(1) \frac{1}{\Psi_k(w,w')} \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon). \end{aligned}$$

Therefore

$$\begin{aligned} &\iint_{\mathcal{B}} \Big[\mathtt{q}_k(t_1,t_2,t_2,w,w') - \mathtt{q}_k\big(\mathtt{t}^{\text{int}}(t_1,w,w') - \varepsilon, w, w'\big)\Big] dw dw' \\ &\qquad\qquad \leq \mathcal{O}(1) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \iint_{\mathcal{B}} \frac{d\tau d\tau'}{\Psi_k(w,w')} \\ &\qquad\qquad \leq \mathcal{O}(1) \mathcal{L}^1\Big(\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)\Big) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \\ &\qquad\qquad \leq \mathcal{O}(1) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon), \end{aligned}$$

where the inequality

$$\iint_{\mathcal{B}} \frac{d\tau d\tau'}{\Psi_k(w,w')} \leq \mathcal{L}^1\Big(\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)\Big)$$

is obtained splitting the integral over $\mathcal{B}$ as a sum of integrals over each single interaction point, changing variable as in the proof of Theorem 3.61 and integrating by parts.

Step 4. It holds

$$\iint_{\mathcal{B}} \mathfrak{q}_k\Big(\mathtt{t}^{\mathrm{int}}\Big(t_1,w,w'\Big) - \varepsilon, w, w'\Big) d\tau d\tau' \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big).$$

It holds

$$\begin{aligned}
&\iint_{\mathcal{B}} \mathfrak{q}_k\big(\mathtt{t}^{\mathrm{int}}(t_1,w,w') - \varepsilon, w, w'\big) dw dw' \\
&\quad = \sum_{i=i_1+1}^{i_2} \iint_{\mathcal{B}^i} \mathfrak{q}_k\big((i-1)\varepsilon, w, w'\big) dw dw' \\
&\quad \text{(see (3.53))} \\
&\quad \leq \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \iint_{J^L_m \times J^R_m} \mathfrak{q}((i-1)\varepsilon) dw dw' \\
&\quad \text{(using (3.54)-(3.55) and the fact that for waves } w, w' \text{ interacting at time } i\varepsilon,\ \mathfrak{q}(i\varepsilon, w, w') = 0) \\
&\quad \leq \sum_{i=i_1+1}^{i_2} \Big(\mathfrak{Q}((i-1)\varepsilon) - \mathfrak{Q}(i\varepsilon)\Big) + \mathcal{O}(1)\text{Tot.Var.}(\bar{u}) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \\
&\quad \text{(since } Q^{\mathrm{known}} \text{ is decreasing in time)} \\
&\quad \leq \sum_{i=i_1+1}^{i_2} \Big(\mathfrak{Q}((i-1)\varepsilon) - \mathfrak{Q}(i\varepsilon)\Big) + C\Big(Q^{\mathrm{known}}((i-1)\varepsilon) - Q^{\mathrm{known}}(i\varepsilon)\Big) \\
&\qquad\qquad + \mathcal{O}(1)\text{Tot.Var.}(\bar{u}) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \\
&\quad \text{(by the definition of } \Upsilon \text{ and Corollary 3.59)} \\
&\quad \leq \mathcal{O}(1) \sum_{i=i_1+1}^{i_2} \Big(\Upsilon((i-1)\varepsilon) - \Upsilon(i\varepsilon)\Big) \\
&\quad = \mathcal{O}(1)\Big(\Upsilon(t_1) - \Upsilon(t_2)\Big).
\end{aligned}$$

The conclusion of the proof of the Proposition is now an immediate consequence of the previous four steps, Corollary 3.59 and Proposition 4.13. $\square$

CHAPTER 5

Lagrangian representation for conservation laws

In this chapter we present the third and last result of this thesis, namely the existence of a *Lagrangian representation* for the solution of the Cauchy problem

$$\begin{cases} u_t + F(u)_x = 0, \\ u(t=0) = \bar{u}. \end{cases} \tag{5.1}$$

Here the system is strictly hyperbolic and no GNL/LD assumption is made on the characteristic fields. In particular we will define the notion of Lagrangian representation for conservation laws (see Definition 5.21) and we will prove that for any weak admissible solution of (5.1), with sufficiently small total variation, there exists a Lagrangian representation (see Theorem C in the Introduction, which will be stated again, for completeness, in Section 5.2).

We have already explained in the Introduction what we mean by *Lagrangian representation* of the solution of the Cauchy problem (5.1); why we use the term "Lagrangian representation" and what is its relation with the theory of the linear transport and of the gas dynamics; why it is interesting to construct such a representation. Therefore, we refer to the Introduction for an extensive discussion about these topics.

What we wish to stress here once again is that this chapter is, in some sense, a *work in progress*. Indeed, many further results and corollaries about the Lagrangian representation and the structure of the solution u could be obtained with little effort using the tools we develop in the next sections. However, due to time constraints, we do not insert such results in this thesis. An extensive discussion of the matter will appear in [**BM15a**].

Structure of the chapter. The chapter is organized as follows.

For a scalar equation, $N = 1$, the notion of Lagrangian representation can be introduced with little effort, as we did in Definition 3 in the Introduction. On the contrary, this is not any more the case in the vector setting. Indeed, if $N > 1$, then the density function $\rho(t, w)$ (see again Definition 3 in the Introduction or Definition 5.21 below) should take values in $\mathbb{R}^N$, being, in some sense, the derivative $D_x u(t, \cdot)$ of the solution u. In Section 5.1 we introduce a tool, the *enumeration of waves*, which allows to recover a vector density $\rho(t, w) r(t, w)$ starting from a scalar density $\rho(t, w)$. Here $r(t, w)$ is a unitary vector, defined, for waves of the k-th family, as the k-th generalized eigenvector $\tilde{r}_k$ (see Section 2.1.1) evaluated on the point $\hat{\gamma}(t, w)$ of a curve $w \mapsto \hat{\gamma}(t, w)$, uniquely determined by the position function $\mathtt{x}$ and the scalar density function ρ.

Using the notion of enumeration of waves introduced in Section 5.1 we give in Section 5.2 the precise definition of *Lagrangian representation* for a solution to the N-dimensional system (5.1), we state the main theorem of this Chapter, namely Theorem C and we give a sketch of its proof.

All the other sections of this chapter are devoted to prove Theorem C. In particular, in Section 5.3 we prove some local interaction estimate, in the same spirit as what we did in Section 3.2 for two merging Riemann problems and in Section 4.3 for the interactions among many colliding Riemann problem (with all the interacting wavefronts of the same family having the same sign). Here, on the contrary, we will consider the situation where there are many interacting Riemann problems, with wavefronts having different signs, but we are interested only in estimating how far the outgoing Riemann problem is from a contact discontinuity, in terms of the distance of the incoming Riemann problems from a contact discontinuity.

In Section 5.4 we continue the analysis started in Chapter 3 on an approximate solution u^ε constructed by means of the Glimm scheme. In particular we will focus on those result, namely the existence of an approximate position map $\mathtt{x}^\varepsilon(t,w)$, an approximate density function $\rho^\varepsilon(t,w)$ and an interaction measure μ^ε, which will be used in the following sections to conclude the proof of Theorem C.

Sections 5.6 and 5.7 are the heart of the proof of Theorem C. In Section 5.6 we first construct the position map $\mathtt{x}(t,w)$ and the density $\rho(t,w)$, our candidate Lagrangian representation for the solution u of (5.1). Then we study the convergence of the curves $\hat{\gamma}^\varepsilon$ constructed through the techniques of Section 5.1 starting from the approximate position $\mathtt{x}^\varepsilon$ and approximate density ρ^ε to the curve $\hat{\gamma}$ constructed starting from $\mathtt{x}(t,w)$ and $\rho(t,w)$. This is probably the most important section of this chapter, where it is basically proved that the notion of Lagrangian representation is stable w.r.t. the L^1 convergence of the solutions.

Finally in Section 5.7 we conclude the proof of Theorem C, proving the our candidate Lagrangian representation $\mathtt{x}(t,w)$, $\rho(t,w)$ is actually a Lagrangian representation in the sense of Definition 5.21.

5.1. Enumeration of waves and related objects

In this section we define the notions of *enumeration of waves* and we construct some related objects. Roughly speaking, an enumeration of waves is a pair of functions $\mathtt{x}, \rho$ defined on a set $\mathcal{W} \subseteq \mathbb{R}$ called the *set of waves*, which describe, respectively, the *position* $\mathtt{x}(w)$ of a wave w and the *density* $\rho(w)$ of a wave w. Given an enumeration of waves, we will show how we can construct many objects, and in particular in Section 5.1.5 a curve $\hat{\gamma}$ in $\mathbb{R}^{3N}$, which will be the main tool to relate the notion of enumeration of waves to the solution of the Cauchy problem (5.1). What is fundamental to stress here is that given any $\mathtt{x}$ and ρ (without any relation with the solution of the Cauchy problem (5.1)) we describe an algorithm to construct the curve $\hat{\gamma}$, which is uniquely determined by $\mathtt{x}$ and ρ. As we pointed out at the beginning of the Chapter, what we have in mind is to give a procedure which allows to reconstruct the solution $u(t,\cdot) = S_t\bar{u}$ to the Cauchy problem (5.1) starting from the position map $\mathtt{x}$ and the density map ρ.

5.1.1. Definition of enumeration of waves.

Our analysis starts with the definition of the notion of *enumeration of waves* and the construction of some auxiliary objects which are needed to prove the existence and uniqueness of the curve γ in the next section.

DEFINITION 5.1. An *enumeration of waves* is a $(N+4)$-tuple $(L_0, \dots, L_N, \mathtt{x}, \rho, \bar{\rho})$ satisfying the following conditions:

1. $L_0 \leq \cdots \leq L_N$; the set $(L_0, L_N]$ is called the *set of waves*, while, for every $k = 1, \dots, N$, the set $(L_{k-1}, L_k]$ is called the *set of k-th waves*;
2. $\mathtt{x} : (L_0, L_N] \to \mathbb{R}$ is called the *position function* and it is an increasing left-continuous map on each subinterval $(L_{k-1}, L_k]$, for $k = 1, \dots, N$;
3. $\rho : (L_0, L_N] \to [-1, 1]$ is a measurable map called the *density function*;
4. $\bar{\rho} : (L_0, L_N] \to [0, 1]$ is a measurable map called the *absolute density function*;
5. it holds $|\rho| \leq \bar{\rho}$.

REMARK 5.2. It will be shown in the following that the presence of two different functions ρ, $\bar{\rho}$ is due to the lower semi-continuity of the weak convergence.

We define now some additional objects, related to the notion of enumeration of waves, which will be frequently used in the paper. Assume that an enumeration of waves

$$(L_0, \dots, L_N, \mathtt{x}, \rho, \bar{\rho})$$

is given.
First of all denote by $\mathtt{k}(w)$ the family of a wave $w \in (L_0, L_N]$, i.e. $\mathtt{k}(w) = \bar{k}$ if $w \in (L_{\bar{k}-1}, L_{\bar{k}}]$. For every $x \in \mathbb{R}$, define the k-*sign* of the point x as

$$\mathcal{S}_k(x) := \operatorname{sign}\bigg(\int_{\mathtt{x}^{-1}(x) \cap (L_{k-1}, L_k]} \rho(w) dw \bigg), \tag{5.2}$$

where, by convention, $\operatorname{sign}(0) = 0$.

5.1.2. The order relation $\Re$. In this section we define an order relation $\Re$ on the set of waves $(L_0, L_N]$ and we prove some of its properties. The relation $\Re$ allows to compare waves which belong to different families, according to their relative position.

For every $w, w' \in (0, L_N]$, set

$$w \Re w' \iff \begin{cases} \text{either } \mathtt{x}(w) < \mathtt{x}(w'), \\ \text{or } \mathtt{x}(w) = \mathtt{x}(w') \text{ and } \mathtt{k}(w) < \mathtt{k}(w'), \\ \text{or } \mathtt{x}(w) = \mathtt{x}(w') \text{ and } \mathtt{k}(w) = \mathtt{k}(w') \text{ and } w \leq w'. \end{cases} \tag{5.3}$$

It is fairly easy to see that $\Re$ is a total order on the set of waves $(L_0, L_N]$. Being $\mathtt{x}|_{(L_{k-1}, L_k]}$ increasing, the relation $\Re$ coincide with $\leq$ for the waves belonging to the same family.

We first prove a regularity property of the set $\Re$, namely that it is a set of finite perimeter.

LEMMA 5.3. *For every $k, h \in \{1, \dots, N\}$, $h \leq k$, $\Re \cap ((L_{h-1}, L_h] \times (L_{k-1}, L_k])$ is the epigraph of an increasing map $(L_{h-1}, L_h] \to (L_{k-1}, L_k]$, up to $\mathcal{L}^2$-negligible sets.*

PROOF. It is easy to see that, up to $\mathcal{L}^2$-negligible sets,

$$w \Re y \iff y \geq \inf \big\{ y' \in (L_{h-1}, L_h] \;\big|\; w \;\Re\; y' \big\}$$

and the map

$$w \mapsto \inf \big\{ y' \in (L_{h-1}, L_h] \;\big|\; w \;\Re\; y' \big\}$$

is increasing. $\square$

Since for $h \neq k$

$$\Re \cap \big((L_{k-1}, L_k] \times (L_{h-1}, L_h] \big) = \big((L_{h-1}, L_h] \times (L_{k-1}, L_k] \big) \setminus \Re,$$

we deduce the following result.

COROLLARY 5.4. *The map $\chi_\Re$ is BV on the open set $(L_0, L_N)^2$ and its total variation is bounded by $4N(L_N - L_0)$.*

In fact, from the monotonicity

$$\mathcal{H}^1 \Big(\partial \Re \cap \big((L_{k-1}, L_k) \times (L_{h-1}, L_h) \big) \Big) \leq L_k - L_{k-1} + L_h - L_{h-1}.$$

The next lemma substitutes the existence of the sup and the inf of sets w.r.t. the order $\Re$ and it will be a useful technical tool later. It basically follows form the fact that the order $\Re$ is in some sense a lexicographic product of the standard order on $\mathbb{R}$.

LEMMA 5.5. *Let $E \subseteq (0, L]$. Then there exist two countable sequences $(w'_r)_r$, $(w''_r)_r$ in E such that (w'_r) is decreasing w.r.t. the order $\Re$, (w''_r) is increasing w.r.t. the order $\Re$ and for every $w \in E$, there is $\bar{r}$ such that for every $r \geq \bar{r}$*

$$w'_r \;\Re\; w \;\Re\; w''_r.$$

We will call $(w'_r)_r$ a *minimizing sequence for* E and $(w''_r)_r$ a *maximizing sequence for* E.

PROOF. We prove only the existence of the sequence (w'_r); the existence of (w''_r) can be proved in an analogous way. Consider the set

$$\mathtt{x}(E) := \{\mathtt{x}(w) \mid w \in E\}.$$

Distinguish two cases:

(1) if $\min \mathtt{x}(E)$ does not exist, take a decreasing minimizing sequence $(x'_r)_{r\in\mathbb{N}}$ in $\mathtt{x}(E)$ and for each $r \in \mathbb{N}$, define w'_r as any element of $\mathtt{x}^{-1}(x'_r)$;
(2) if $\bar{x} := \min \mathtt{x}(E)$ exists, then distinguish two more cases:
  (a) if $\min \mathtt{x}^{-1}(\bar{x})$ does not exists, then define (w'_r) as any decreasing minimizing sequence in $\mathtt{x}^{-1}(\bar{x})$;
  (b) if $\min \mathtt{x}^{-1}(\bar{x})$ exists, then define $w'_r := \min \mathtt{x}^{-1}(\bar{x})$ for every $r \in \mathbb{N}$.

It is easily shown that (w'_r) satisfies the property in the statement of the lemma. The construction of (w''_r) is analogous. □

5.1.3. The functions V, V_k, ω_k. In the e.o.w. the quantity $\bar{\rho}(w)$ represents the absolute density of the wave w. In this section we introduce a map V (resp. a map V_k) which set a correspondence between the given set of waves $(L_0, L_N]$ (resp. $(L_{k-1}, L_k]$, $k = 1, \dots, N$) and an artificial set of waves $(0, M]$ (resp. $(0, M_k]$, $k = 1, \dots, N$) where all the waves have absolute density equal to 1. This set would be the natural parameterization of the e.o.w. in the time independent case, i.e. if we were not considering also the time evolution of the solution. We introduce also a map ω_k which relates the elements of $(0, M_k]$ with the elements of $(0, M]$, according to the order $\Re$.

First of all set

$$M_k := \int_{L_{k-1}}^{L_k} \bar{\rho}(w)dw, \qquad M := \sum_{k=1}^{N} M_k = \int_{L_0}^{L_N} \bar{\rho}(w)dw. \tag{5.4}$$

Define now the following maps: for $k = 1, \dots, N$, let

$$V_k : (L_{k-1}, L_k] \to (0, M_k], \quad V_k(z) := \int_{L_{k-1}}^{L_k} \chi_{\Re}(y, w)\bar{\rho}(y)dy = \int_{L_{k-1}}^{w} \bar{\rho}(y)dy, \tag{5.5a}$$

and

$$V : (L_0, L_N] \to (0, M], \qquad V(w) := \int_{L_0}^{L_N} \chi_{\Re}(y, w)\bar{\rho}(y)dy. \tag{5.5b}$$

Clearly V_k and $V|_{(L_{k-1},L_k]}$ are increasing for every $k = 1, \dots, N$, and in the sense of measure

$$0 \le D_w V_k = \bar{\rho}\mathcal{L}^1 \llcorner_{(L_{k-1},L_k]} \le D_w V \llcorner_{(L_{k-1},L_k]}. \tag{5.6}$$

The numbers M, M_k, $k = 1, \dots, N$ express the global amount of waves, each one summed with its own absolute density.

The following lemma is a technical tool which will be used later.

LEMMA 5.6. *For every $z \in [0, M]$ it holds*

$$\int_{V^{-1}((0,z])} \bar{\rho}(w)dw = z.$$

PROOF. Set $E := V^{-1}\big((0, z]\big)$. By previous lemma, there exist a maximizing sequence $(w''_r)_r$ for E. It is easy to see that

$$E = \bigcup_{r\in\mathbb{N}} E''_r$$

where $E''_r := \{w \in E \mid w \Re w''_r\}$. Since (w''_r) is increasing w.r.t. the order $\Re$, the sequence $(E''_r)_r$ is increasing w.r.t. the set inclusion. Thus it holds

$$\int_E \bar{\rho}(w)dw = \lim_{r\to\infty} \int_{E''_r} \bar{\rho}(w)dw = \lim_{r\to\infty} V(w''_r) \leq z,$$

because for every $r \in \mathbb{N}$, $w_r \in E = V^{-1}\big((0,z]\big)$.
Similarly, let $(w'_r)_r$ be a minimizing sequence for $(0,L] \setminus E$. One can easily prove that

$$E = \bigcap_{r\in\mathbb{N}} E'_r$$

where $E'_r := \{w \in E \mid w \Re w'_r\}$. Since (w'_r) is decreasing w.r.t. the order $\Re$, the sequence $(E'_r)_r$ is decreasing w.r.t. the set inclusion. Thus we have

$$\int_E \bar{\rho}(w)dw = \lim_{r\to\infty} \int_{E'_r} \bar{\rho}(w)dw = \lim_r V(w'_r) \geq z,$$

because for every r, $w''_r \notin E = V^{-1}\big((0,L]\big)$. Hence

$$z \leq \int_E \bar{\rho}(w)dw \leq z,$$

thus concluding the proof of the lemma. □

Corollary 5.7. *For every $k = 1, \dots, N$,*

$$(V_k)_\sharp\big(\bar{\rho}\mathcal{L}^1|_{(L_{k-1},L_k]}\big) = \mathcal{L}^1|_{(0,M_k]}$$

and

$$V_\sharp\big(\bar{\rho}\mathcal{L}^1|_{(L_0,L_N]}\big) = \mathcal{L}^1|_{(0,M]}.$$

Proof. The proof of the first equality is trivial. The second equality follows from Lemma 5.6. □

The following proposition states some properties of the maps V, V_k, $k = 1, \dots, N$.

Proposition 5.8. *The following properties hold.*

(1) V_k is 1-Lipschitz, increasing and surjective.
(2) V is increasing w.r.t. the order $\Re$ and for every $k = 1, \dots, N$, the restriction $V|_{(L_{k-1},L_k]}$ is increasing w.r.t. the standard order $\leq$ on $\mathbb{R}$.
(3) V is surjective.

Proof. Point (1) and (2) are straightforward. Let us prove Point (3). Let $z \in (0,M]$. Assume by contradiction that there is no $w \in (L_0, L_N]$ such that $V(w) = z$. Since each restriction $V|_{(L_{k-1},L_k]}$ is increasing, there must be $\delta > 0$ such that

$$V^{-1}\big((z-\delta, z+\delta]\big) = \emptyset.$$

Hence

$$\begin{aligned} 0 &= \int_{V^{-1}((z-\delta,z+\delta])} \bar{\rho}(w)dw \\ &= \int_{V^{-1}((0,z+\delta])} \bar{\rho}(w)dw - \int_{V^{-1}((0,z-\delta])} \bar{\rho}(w)dw \\ \text{(by Lemma 5.6)} \quad &= (z+\delta) - (z-\delta) \\ &= 2\delta, \end{aligned}$$

a contradiction. □

Define now for every $k = 1, \dots, N$, the map

$$\omega_k : (0, M_k] \to (0, M]$$

as the unique left-continuous map such that

$$\omega_k(z) = V(V_k^{-1}(z)) \text{ for every } z \in (0, M_k] \text{ such that } V_k^{-1}(z) \text{ is single-valued.} \tag{5.7}$$

The following proposition collects some properties of ω_k.

PROPOSITION 5.9. *For every $k = 1, \dots, N$,*

(1) The definition of ω_k is well posed.
(2) The map ω_k is strictly increasing (and thus injective).
(3) For every $w \in (L_{k-1}, L_k]$ such that $V_k^{-1}(V_k(w))$ is single-valued (and thus for $\bar{\rho}\mathcal{L}^1$-a.e. $w \in (L_{k-1}, L_k]$), it holds $V(w) = \omega_k(V_k(w))$.

PROOF. We prove separately each point.

(1) Since V_k is increasing, there exists at most countable many $z \in (0, M_k]$ such that $V_k^{-1}(z)$ is not single-valued, and thus the definition of ω_k is well posed.
(2) The map ω_k is increasing, since it is a composition of two increasing maps. Assume now by contradiction that ω_k is not strictly increasing. Then there are $z, z' \in (0, M_k]$, $z < z'$ such that $\omega_k(z) = \omega_k(z') = \zeta$. Since we already know that ω_k is increasing, there exists an open interval $(a, b) \subseteq \omega_k^{-1}(\zeta)$ and thus we can find $\tilde{z}, \tilde{z}' \in (a, b) \subseteq \omega_k^{-1}(\zeta)$, $\tilde{z} < \tilde{z}'$, such that $V_k^{-1}(\tilde{z})$, $V_k^{-1}(\tilde{z}')$ are single valued. We can thus set $w := V_k^{-1}(\tilde{z})$, $w' := V_k^{-1}(\tilde{z}')$ and it holds $w < w'$. Hence
$$\omega_k(\tilde{z}) = V(V_k^{-1}(\tilde{z})) = V(w) = \zeta = \omega_k(\tilde{z}') = V(V_k^{-1}(\tilde{z}')) = V(w').$$
Therefore
$$0 = \int_{\{y \in (L_0, L_N] \mid w \Re y \text{ and } y \Re w'\}} \bar{\rho}(y)dy \geq \int_w^{w'} \bar{\rho}(y)dy$$
and thus $\tilde{z} = V_k(w) = V_k(w') = \tilde{z}'$, a contradiction.
(3) The proof follows easily from the definition of ω_k. □

5.1.4. The position functions $\hat{\mathtt{x}}$, $\hat{\mathtt{x}}_k$. In this section we extend the definition of the position function to the set of artificial waves $(0, M]$, $(0, M_k]$, $k = 1, \dots, N$ introduced in the previous section, and we prove some properties of these new position functions.

Define

$$\hat{\mathtt{x}}_k : (0, M_k] \to \mathbb{R},$$

as the unique left-continuous maps such that

$$\hat{\mathtt{x}}_k(z) = \mathtt{x}\big(V_k^{-1}(z)\big) \text{ for every } z \in (0, M_k] \text{ such that } V_k^{-1}(z) \text{ is single valued.} \tag{5.8}$$

To define a similar map $\mathtt{x} : (0, M] \to \mathbb{R}$, we need the following lemma.

LEMMA 5.10. *The multi-valued map*

$$z \mapsto \mathtt{x}\big(V^{-1}(z)\big)$$

is increasing.

See Section 1.5 for the definition and the properties of monotone multi-valued functions.

PROOF. Since for every k, $V|_{(L_{k-1},L_k]}$ is increasing, we have that for a.e. $\zeta \in (0,M]$, $\operatorname{card} V^{-1}(\zeta) \le N$. Let $z, z' \in (0,M]$, $z < z'$ such that

$$V^{-1}(z) = \{w_1, \dots, w_N\}, \qquad V^{-1}(z') = \{w'_1, \dots, w'_N\}$$

(we assume for simplicity that there are exactly N elements in $V^{-1}(z)$ and $V^{-1}(z')$). Hence

$$\mathtt{x}\big(V^{-1}(z)\big) = \big\{\mathtt{x}(w_1), \dots, \mathtt{x}(w_N)\big\}, \qquad \mathtt{x}\big(V^{-1}(z')\big) = \big\{\mathtt{x}(w'_1), \dots, \mathtt{x}(w'_N)\big\}.$$

We want to prove that for every $k, h \in \{1, \dots, N\}$, $\mathtt{x}(w_k) \le \mathtt{x}(w'_h)$. Assume by contradiction that $\mathtt{x}(w_k) > \mathtt{x}(w'_h)$. This implies that $w'_h \Re w_k$ and thus $z' = V(w'_h) \le V(v_k) = z$, a contradiction. □

As an immediate consequence of previous lemma and Lemma 1.41 we can now define

$$\hat{\mathtt{x}} : (0,M] \to \mathbb{R},$$

as the unique left-continuous maps such that

$$\hat{\mathtt{x}}(z) = \mathtt{x}\big(V^{-1}(z)\big) \text{ up to countable set of } z \in (0,M].$$

The following proposition collects some properties of the maps $\hat{\mathtt{x}}$, $\hat{\mathtt{x}}_k$. Its proof is analog to the proof of Proposition 5.9 and thus it is omitted.

PROPOSITION 5.11. *The following hold.*

(1) The definitions of $\hat{\mathtt{x}}_k, \hat{\mathtt{x}}$ are well posed.
(2) The maps $\hat{\mathtt{x}}_k, \hat{\mathtt{x}}$ are increasing.
(3) For $\mathcal{L}^1$-a.e. $w \in (L_{k-1}, L_k]$ such that $V_k^{-1}(V_k(w))$ is single-valued (and thus for $\bar{\rho}\mathcal{L}^1$-a.e. $w \in (L_{k-1}, L_k]$), it holds

$$\mathtt{x}(w) = \hat{\mathtt{x}}_k(V_k(w)).$$

(4) For $\bar{\rho}\mathcal{L}^1$-a.e. $w \in (L_0, L_N]$, it holds

$$\mathtt{x}(w) = \hat{\mathtt{x}}(V(w)).$$

(5) For every $k = 1, \dots, N$ and for a.e. $z \in (0, M_k]$, it holds

$$\hat{\mathtt{x}}\big(\omega_k(z)\big) = \hat{\mathtt{x}}_k(z).$$

We now explain more in details the relations between the position functions $\hat{\mathtt{x}}$, $\hat{\mathtt{x}}_k$, defined respectively on $(0,M]$, $(0,M_k]$, and the position $\mathtt{x}$ and the absolute density $\bar{\rho}$ defined on $(L_0, L_N]$.

LEMMA 5.12. *For every $k = 1, \dots, N$,*

$$\mathcal{L}^1\big(\hat{\mathtt{x}}_k^{-1}(x)\big) = \int_{\mathtt{x}^{-1}(x)\cap(L_{k-1},L_k]} \bar{\rho}(w)dw.$$

PROOF. Using Corollary 5.7 and Proposition 5.11, we get

$$\mathcal{L}^1\big(\hat{\mathtt{x}}_k^{-1}(x)\big) = \int_{V_k^{-1}(\hat{\mathtt{x}}_k^{-1}(x))} \bar{\rho}(w)dw = \int_{\mathtt{x}^{-1}(x)\cap(L_{k-1},L_k]} \bar{\rho}(w)dw. \qquad \square$$

COROLLARY 5.13. *For every $x \in \mathbb{R}$, it holds*

$$\mathcal{L}^1\big(\hat{\mathtt{x}}^{-1}(x)\big) = \sum_{h=1}^N \mathcal{L}^1\big(\hat{\mathtt{x}}_h^{-1}(x)\big) = \int_{\mathtt{x}^{-1}(x)} \bar{\rho}(w)dw.$$

PROOF. Using Corollary 5.7 and Proposition 5.11, we get

$$\mathcal{L}^1\big(\hat{\mathtt{x}}^{-1}(x)\big) = \int_{V^{-1}(\hat{\mathtt{x}}^{-1}(x))} \bar{\rho}(w)dw = \int_{\mathtt{x}^{-1}(x)} \bar{\rho}(w)dw = \sum_{h=1}^{N} \mathcal{L}^1\big(\hat{\mathtt{x}}_h^{-1}(x)\big). \qquad \square$$

The following two lemmas describe the behavior of the map ω_k, which relates the z-waves of a fixed k-family to the global z-waves.

LEMMA 5.14. *For every $x \in \mathbb{R}$ and for every $k = 1, \dots, N$, if*

$$\mathcal{L}^1\big(\hat{\mathtt{x}}_k^{-1}(x)\big) > 0, \tag{5.9}$$

then the map

$$\omega_k|_{\hat{\mathtt{x}}_k^{-1}(x)} : \hat{\mathtt{x}}_k^{-1}(x) \to \hat{\mathtt{x}}^{-1}(x)$$

is affine with slope equal to 1.

PROOF. Clearly it is enough to prove that for $\mathcal{L}^2$-a.e. $(z, z') \in \big(\hat{\mathtt{x}}_k^{-1}(x)\big)^2$, it holds

$$\omega_k(z') - \omega_k(z) = z' - z.$$

Take thus $z, z' \in (0, M_k]$, $z < z'$, such that $V_k^{-1}(z)$ and $V_k^{-1}(z')$ are single-valued. Set $w := V_k^{-1}(z)$ and $w' := V_k^{-1}(z')$. Since $\hat{\mathtt{x}}_k(z) = \hat{\mathtt{x}}_k(z') = x$, then by definition $\mathtt{x}(w) = \mathtt{x}(w') = x$. Therefore

$$\big\{y \in (L_0, L_N] \;\big|\; w\Re y \text{ and } y\Re w'\big\} = [w, w']$$

and thus

$$\begin{aligned} \omega_k(z') - \omega_k(z) &= V(w') - V(w) \\ &= \int_{\{y\in(L_0,L_N] \;|\; w\Re y \text{ and } y\Re w')\}} \bar{\rho}(y)dy \\ &= \int_w^{w'} \bar{\rho}(y)dy \\ &= V_k(w') - V_k(w) \\ &= z' - z, \end{aligned}$$

proving that $\omega_k|_{\hat{\mathtt{x}}_k^{-1}(x)}$ is affine with slope 1. $\square$

LEMMA 5.15. *For every $x \in \mathbb{R}$, let k_p, $p = 1, \dots, P$ be the indices such that*

$$\mathcal{L}^1\big(\hat{\mathtt{x}}_{k_p}^{-1}(x)\big) > 0,$$

labeled according $k_p < k_{p'}$ for $p < p'$. Then

$$\inf\Big\{\omega_{k_1}\big(\hat{\mathtt{x}}_{k_1}^{-1}(x)\big)\Big\} = \inf \hat{\mathtt{x}}^{-1}(x), \qquad \sup\Big\{\omega_{k_P}\big(\hat{\mathtt{x}}_{k_P}^{-1}(x)\big)\Big\} = \sup \hat{\mathtt{x}}^{-1}(x)$$

and for every $p = 1, \dots, P-1$

$$\sup\Big\{\omega_{k_p}\big(\hat{\mathtt{x}}_{k_p}^{-1}(x)\big)\Big\} = \inf\Big\{\omega_{k_{p+1}}\big(\hat{\mathtt{x}}_{k_{p+1}}^{-1}(x)\big)\Big\}.$$

PROOF. We first prove that for every $p < p'$ and

$$\text{for a.e. } z \in \mathtt{x}_{k_p}^{-1}(x), z' \in \hat{\mathtt{x}}_{k_{p'}}^{-1}(x), \text{ it holds } \omega_{k_p}(z) \le \omega_{k_{p'}}(z'). \tag{5.10}$$

We can assume that $V_{k_p}^{-1}(z)$ and $V_{k_{p'}}^{-1}(z')$ are single-valued, because this condition is verified for a.e. z, z'. Set $w := V_{k_p}^{-1}(z)$ and $w' := V_{k_{p'}}^{-1}(z')$. In this case $\omega_{k_p}(z) = V(w)$, $\omega_{k_{p'}}(z') = V(w')$. Moreover, using Proposition 5.11, we have that

$$\mathtt{x}(w) = \hat{\mathtt{x}}_{k_p}\big(V_{k_p}(w)\big) = \hat{\mathtt{x}}_{k_p}(z) = x = \hat{\mathtt{x}}_{k_{p'}}(z') = \hat{\mathtt{x}}_{k_{p'}}\big(V_{k_{p'}}(w')\big) = \mathtt{x}(w').$$

Therefore w, w' have the same position and thus, by the definition of $\Re$, $w \Re w'$. Hence $\omega_{k_p}(z) = V(w) \leq V(w') = \omega_{k_{p'}}(z')$.

Observe now that the condition (5.10) holds for every z, z' (not just for a.e. z, z'), since, by Lemma 5.14 the maps ω_k are continuous. The conclusion is now an immediate consequence of Corollary 5.13. $\square$

We conclude this section with two technical lemma which will be used in the proof of Lemma 5.68. Introduce first the following sets:

$$S := \Big\{ z \in (0, M] \;\big|\; \hat{\mathtt{x}}^{-1}(\hat{\mathtt{x}}(z)) = \{z\} \Big\},$$

$$S_k := \Big\{ z \in (0, M_k] \;\big|\; \hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z)) = \{z\} \Big\}, \text{ for } k = 1, \dots, N.$$

Lemma 5.16. *The following holds.*

1. *For every $z \in (0, M]$, setting $x := \hat{\mathtt{x}}(z)$,*
$$z \in S \iff \int_{\mathtt{x}^{-1}(x)} \bar{\rho}(w) dw = 0.$$
2. *For every $z \in (0, M_k]$, setting $x := \hat{\mathtt{x}}_k(z)$,*
$$z \in S_k \iff \int_{\mathtt{x}^{-1}(x) \cap (L_{k-1}, L_k]} \bar{\rho}(w) dw = 0.$$
3. *For every $k = 1, \dots, N$, and for every $z \in (0, M_k]$ up to a countable set, if $z \in S_k$, then $\omega_k(z) \in S$.*

Proof. We prove each point separately.

(1) Let $z \in (0, M]$. Set $x := \hat{\mathtt{x}}(z)$. By Corollary 5.13,
$$\mathcal{L}^1\big(\hat{\mathtt{x}}^{-1}(x)\big) = \int_{V^{-1}(\hat{\mathtt{x}}^{-1}(x))} \bar{\rho}(w) dw = \int_{\mathtt{x}^{-1}(x)} \bar{\rho}(w) dw,$$
and thus $z \in S$ if and only if $\int_{\mathtt{x}^{-1}(x)} \bar{\rho}(w) dw = 0$.

(2) The proof is completely similar to Point (1).

(3) Fix a family k. For every $h \neq k$, define
$$E_h := \Big\{ x \in \mathbb{R} \;\Big|\; \mathcal{L}^1\big(\hat{\mathtt{x}}_h^{-1}(x)\big) > 0 \Big\}.$$
Clearly, E_h is countable and thus also $E := \bigcup_{h \neq k} E_h$ is countable. Since $\hat{\mathtt{x}}_k$ is injective on S_k, also $\hat{\mathtt{x}}_k^{-1}(E)$ is countable. It is thus enough to prove that for every $z \in S_k \setminus \hat{\mathtt{x}}_k^{-1}(E)$ such that $\hat{\mathtt{x}}(\omega_k(z)) = \hat{\mathtt{x}}_k(z)$, it holds $\omega_k(z) \in S$. Take thus any $z \in S_k \setminus \hat{\mathtt{x}}_k^{-1}(E)$ such that $\hat{\mathtt{x}}(\omega_k(z)) = \hat{\mathtt{x}}_k(z)$. Define $x := \hat{\mathtt{x}}_k(z)$. By Corollary 5.13
$$\mathcal{L}^1\big(\hat{\mathtt{x}}^{-1}(x)\big) = \sum_{h=1}^{N} \mathcal{L}^1\big(\hat{\mathtt{x}}_h^{-1}(x)\big) = 0,$$

because $z \in S_k \setminus \hat{\mathtt{x}}_k^{-1}(E)$. Hence

$$0 = \mathcal{L}^1\Big(\hat{\mathtt{x}}^{-1}\big(\hat{\mathtt{x}}_k(z)\big)\Big) = \mathcal{L}^1\Big(\hat{\mathtt{x}}^{-1}\big(\hat{\mathtt{x}}(\omega_k(z))\big)\Big),$$

and thus $\omega_k(z) \in S$. □

LEMMA 5.17. *For every $k = 1, \dots, N$ and for every $z \in (0, M_k]$ up to a countable set, setting $x := \hat{\mathtt{x}}_k(z)$, if*

$$\int_{\mathtt{x}^{-1}(x)} \bar{\rho}(w)dw = 0, \tag{5.11}$$

then the set

$$V^{-1}\big((0, \omega_k(z)]\big) \triangle x^{-1}\big((-\infty, x]\big)$$

is $\bar{\rho}\mathcal{L}^1$-negligible.

PROOF. Fix a family k and take any $z \in (0, M_k]$ such that $V_k^{-1}(z)$ is single-valued and $V^{-1}(\omega_k(z))$ is single-valued. Clearly this happens for every z up to countable set, by Proposition 5.9, Point (2). In this case, by definition, $\omega_k(z) = V(V_k^{-1}(z))$. Set $x := \hat{\mathtt{x}}_k(z)$ and assume that (5.11) holds. We prove first that for $\bar{\rho}\mathcal{L}^1$-a.e. $w \in V^{-1}((0, \omega_k(z)])$ it holds $w \in \mathtt{x}^{-1}((-\infty, x])$. Take any $w \in V^{-1}((0, \omega_k(z)])$ such that $V^{-1}(V(w))$ is single valued. Then

$$V(w) \leq \omega_k(z)$$

and thus

$$\mathtt{x}(w) = \hat{\mathtt{x}}(V(w)) \leq \hat{\mathtt{x}}(\omega_k(z)) = \hat{\mathtt{x}}\big(V(V_k^{-1}(z))\big) = \mathtt{x}(V_k^{-1}(z)) = \hat{\mathtt{x}}_k(z) = x.$$

Viceversa, assume that $w \in \mathtt{x}^{-1}((-\infty, x])$ and suppose that $V(w) > \omega_k(w) = V(V_k^{-1}(w))$. Then

$$0 = \int_{\mathtt{x}^{-1}(x)} \bar{\rho}(y)dy \geq \int_{\{y \in (L_0, L_N] \;|\; \Re(V_k^{-1}(z), y) \text{ and } \Re(y, w)\}} \bar{\rho}(y)dy > 0,$$

a contradiction. □

5.1.5. Fixed point problem associated to an e.o.w. To a given enumeration of waves $(L_0, \dots, L_N, \mathtt{x}, \rho, \bar{\rho})$ and related objects $\Re, M, M_k, V, V_k, \omega_k, \hat{\mathtt{x}}, \hat{\mathtt{x}}_k$, defined for $k = 1, \dots, N$ as in Sections 5.1.2, 5.1.3, 5.1.4, we now associate a fixed point problem, whose solution is a curve $\gamma = (u, v_1, \dots, v_N, \sigma_1, \dots, \sigma_N) \in \mathbb{R}^{3N}$ which will be the fundamental tool to relate the given e.o.w. to a solution $u(t, x)$ of the Cauchy problem (5.1).
Consider the Banach space

$$X := L^\infty([0, M]; \mathbb{R}^N) \times \prod_{k=1}^N L^\infty([0, M_k]) \times \prod_{k=1}^N L^1([0, M_k]) \tag{5.12}$$

and a generic element

$$\gamma = (u, v_1, \dots, v_N, \sigma_1, \dots, \sigma_N) \in X.$$

The norm on X will be denoted as

$$\|\gamma\|_\dagger := \|u\|_\infty + \sum_{k=1}^N \|v_k\|_\infty + \sum_{k=1}^N \|\sigma_k\|_1.$$

For $\gamma \in X$, set

$$r^\gamma : (L_0, L_N] \to \mathbb{R}^n, \quad r^\gamma(w) := \tilde{r}_k\Big(u\big(V(w)\big), v_k\big(V_k(w)\big), \sigma_k\big(V_k(w)\big)\Big), \text{ if } w \in (L_{k-1}, L_k], \tag{5.13a}$$

$$\lambda^\gamma : (L_0, L_N] \to \mathbb{R}^n, \quad \lambda^\gamma(w) := \tilde{\lambda}_k\Big(u\big(V(w)\big), v_k\big(V_k(w)\big), \sigma_k\big(V_k(w)\big)\Big), \text{ if } w \in (L_{k-1}, L_k], \tag{5.13b}$$

whenever defined. Notice that $\tilde{r}_k, \tilde{\lambda}_k$ are defined on a neighborhood of the point $(0, 0, \lambda_k(0)) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$, and thus r^γ, λ^γ are defined only for the curves $\gamma \in X$ which remain sufficiently close to

$$P := (0, 0, \ldots, 0, \lambda_1(0), \ldots, \lambda_N(0)).$$

Set also, for every $k = 1, \ldots, N$,

$$f_k^\gamma : [0, M_k] \to \mathbb{R}, \qquad f_k^\gamma(z) := \int_{(0,z]} (V_k)_\sharp \Big(\bar{\rho}\lambda^\gamma \mathcal{L}^1|_{(L_{k-1}, L_k]}\Big)(d\zeta). \tag{5.14}$$

The following lemma provides some properties of the maps f_k^γ, $k = 1, \ldots, N$.

LEMMA 5.18. *Assume that $\gamma \in X$ is a curve which remains close enough to P, in order to guarantee that λ^γ and thus f_k^γ are well defined. Then it holds:*

(1) for every $z \in [0, M_k]$,

$$f_k^\gamma(z) = \int_0^z \tilde{\lambda}_k\Big(u\big(\omega_k(\zeta)\big), v_k(\zeta), \sigma_k(\zeta)\Big) d\zeta; \tag{5.15}$$

(2) f_k^γ is Lipschitz,

$$\mathrm{Lip}(f_k^\gamma) \leq \|\tilde{\lambda}_k\|_\infty;$$

and, for a.e. $z \in [0, M_k]$, it holds

$$\frac{df_k^\gamma}{dz}(z) = \tilde{\lambda}_k\big(u\big(\omega_k(z)\big), v_k(z), \sigma_k(z)\big) \text{ for } \mathcal{L}^1\text{-a.e. } z \in [0, M_k]; \tag{5.16}$$

(3) if u, v_k are Lipschitz and σ_k is BV, then $\frac{df_k^\gamma}{dz}$ is BV and

$$\mathrm{e.Tot.Var.}\left(\frac{df_k^\gamma}{dz}\right) \leq \mathcal{O}(1)\left[\Big(\mathrm{Lip}(u) + \mathrm{Lip}(v_k)\Big) M + \left\|\frac{\partial \tilde{\lambda}_k}{\partial \sigma_k}\right\|_\infty \mathrm{e.Tot.Var.}(v_k)\right].$$

PROOF. We prove each point separately.

(1) It holds

$$\begin{aligned}
f_k^\gamma(z) &= \int_{(0,z]} (V_k)_\sharp \Big(\bar{\rho}\lambda^\gamma \mathcal{L}^1|_{(L_{k-1}, L_k]}\Big)(d\zeta) \\
&= \int_{V_k^{-1}((0,z])} \bar{\rho}(w)\lambda^\gamma(w) dw \\
\text{(by (5.13))} \quad &= \int_{V_k^{-1}((0,z])} \bar{\rho}(w)\tilde{\lambda}_k\Big(u\big(V(w)\big), v_k\big(V_k(w)\big), \sigma_k\big(V_k(w)\big)\Big) dw \\
&\quad \text{(using Proposition 5.9, Point (3))} \\
&= \int_{V_k^{-1}((0,z])} \bar{\rho}(w)\tilde{\lambda}_k\Big(u\big(\omega_k(V_k(w))\big), v_k\big(V_k(w)\big), \sigma_k\big(V_k(w)\big)\Big) dw \\
&\quad \text{(making the change of variable } \zeta = V_k(w) \text{ and using that } V_k \text{ is surjective)} \\
&= \int_0^z \tilde{\lambda}_k\big(u\big(\omega_k(\zeta)\big), v_k(\zeta), \sigma_k(\zeta)\big) d\zeta,
\end{aligned}$$

thus getting (5.15).

(2) Since $\zeta \mapsto \tilde{\lambda}_k\big(u\big(\omega_k(\zeta)\big), v_k(\zeta), \sigma_k(\zeta)\big)$ is in L^∞, then f_k^γ is Lipschitz with $\mathrm{Lip}(f_k^\gamma) \leq \|\tilde{\lambda}_k\|_\infty$. Thus, it is a.e. differentiable and (5.16) holds.

(3) Assume now that u, v_k are Lipschitz for every $k = 1, \dots, N$ and σ_k is BV for every $k = 1, \dots, N$. For every $z_1, z_2 \in [0, M_k]$, $z_1 < z_2$, it holds

$$\begin{aligned}
&\Big|\tilde\lambda_k\big(u(\omega_k(z_2)), v_k(z_2), \sigma_k(z_2)\big) - \tilde\lambda_k\big(u(\omega_k(z_1)), v_k(z_1), \sigma_k(z_1)\big)\Big| \\
&\quad\le \left\|\frac{\partial\tilde\lambda_k}{\partial u}\right\|_\infty \Big|u\big(\omega_k(z_2)\big) - u\big(\omega_k(z_1)\big)\Big| + \left\|\frac{\partial\tilde\lambda_k}{\partial v_k}\right\|_\infty \Big|v_k(z_2) - v_k(z_1)\Big| + \left\|\frac{\partial\tilde\lambda_k}{\partial \sigma_k}\right\|_\infty \Big|\sigma_k(z_2) - \sigma_k(z_1)\Big| \\
&\quad\le \left\|\frac{\partial\tilde\lambda_k}{\partial u}\right\|_\infty \mathrm{Lip}(u)\big|\omega_k(z_2) - \omega_k(z_1)\big| + \left\|\frac{\partial\tilde\lambda_k}{\partial v_k}\right\|_\infty \mathrm{Lip}(v_k)\big|z_2 - z_1\big| + \left\|\frac{\partial\tilde\lambda_k}{\partial \sigma_k}\right\|_\infty \Big|\sigma_k(z_2) - \sigma_k(z_1)\Big|.
\end{aligned}$$

Hence, using Proposition 5.9, Point (2), and Corollary 1.38, we get

$$\begin{aligned}
\text{e.Tot.Var.}\left(\frac{df_k^\gamma}{dz}\right) &\le \left\|\frac{\partial\tilde\lambda_k}{\partial u}\right\|_\infty \mathrm{Lip}(u) M + \left\|\frac{\partial\tilde\lambda_k}{\partial v_k}\right\|_\infty \mathrm{Lip}(v_k) M + \left\|\frac{\partial\tilde\lambda_k}{\partial \sigma_k}\right\|_\infty \text{e.Tot.Var.}(\sigma_k) \\
&\le \mathcal{O}(1)\left[\Big(\mathrm{Lip}(u) + \mathrm{Lip}(v_k)\Big)M + \left\|\frac{\partial\tilde\lambda_k}{\partial \sigma_k}\right\|_\infty \text{e.Tot.Var.}(\sigma_k)\right],
\end{aligned}$$

thus concluding the proof of the lemma. □

Our aim is now to find a (unique) curve $\hat\gamma = (\hat u, \hat v_1, \dots, \hat v_N, \hat\sigma_1, \dots, \hat\sigma_N) \in X$, with $\hat u$, $\hat v_k$, Lipschitz and $\hat\sigma_k$ in BV, for $k = 1, \dots, N$, which solves the fixed point problem

$$\begin{cases}
\hat u(z) := \displaystyle\int_{(0,z]} V_\sharp\big(\rho r^{\hat\gamma}\mathcal{L}^1|_{(L_0, L_N]}\big)(d\zeta), \\
\hat v_k(z) := \mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)\Big(f_k^{\hat\gamma}(z) - \displaystyle\operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} f_k^{\hat\gamma}(z)\Big), \qquad k = 1, \dots, N, \\
\hat\sigma_k(z) := \dfrac{d}{dz}\displaystyle\operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} f_k^{\hat\gamma}(z), \qquad k = 1, \dots, N.
\end{cases} \tag{5.17}$$

More precisely, we prove the following proposition.

PROPOSITION 5.19. *There exists $\bar M > 0$ (depending only on f) such that for every enumeration of waves $(L_0, \dots, L_N, \mathtt{x}, \rho, \bar\rho)$ (and related objects M_k, M, V_k, V, ω_k, $\hat{\mathtt{x}}_k$, $\hat{\mathtt{x}}$ defined as in (5.4), (5.5), (5.7), (5.8) above), if $0 < M \le \bar M$, then:*

(1) the fixed point problem (5.17) admits a solution $(\hat u, \hat v_1, \dots, \hat v_N, \hat\sigma_1, \dots \hat\sigma_N)$, with the maps $\hat u, \hat v_1, \dots, \hat v_N$ Lipschitz continuous and $\hat\sigma_1, \dots, \hat\sigma_N$ in BV;

(2) such solution is unique in the class of Lipschitz-Lipschitz-BV functions in the sense that if $\hat\gamma = (\hat u, \hat v_1, \dots, \hat v_N, \hat\sigma_1, \dots, \hat\sigma_N)$ and $\hat\gamma' = (\hat u', \hat v_1', \dots, \hat v_N', \hat\sigma_1', \dots, \hat\sigma_N')$ are solutions of the fixed point problem (5.17), $\hat u, \hat u', \hat v_1, \hat v_1', \dots, \hat v_N, \hat v_N'$ are Lipschitz and $\hat\sigma_1, \hat\sigma_1', \dots, \hat\sigma_N, \hat\sigma_N'$ are BV, then $\hat\gamma = \hat\gamma'$;

(3) if $(\hat u, \hat v_1, \dots, \hat v_N, \hat\sigma_1, \dots \hat\sigma_N)$ is the unique solution of (5.17) given by Points (1)-(2) above, then $\mathrm{Lip}(\hat u), \mathrm{Lip}(\hat v_1), \dots, \mathrm{Lip}(\hat v_N)$ and $\text{e.Tot.Var.}(\hat\sigma_1), \dots, \text{e.Tot.Var.}(\hat\sigma_N)$ are bounded by some constant C which depends only on f and not on $\mathtt{x}, \rho, \bar\rho$.

PROOF. The proof is divided in several steps. In Step 1, 2, 3 we prove the existence of a regular solution to the system (5.17). In particular, in Step 1 we define a closed subset (and thus complete metric space) $\Gamma \subseteq X$ and a map $\mathcal{T} : \Gamma \to X$ such that any fixed point of $\mathcal{T}$ is a solution of the system (5.17) satisfying the regularity properties of Point (1) in the statement. In Step 2 we prove that $\mathcal{T} : \Gamma \to \Gamma$. In Step 3 we prove that $\mathcal{T}$ is a contraction with contractive constant less or equal than $1/2$. In Step 4 we prove the uniqueness property, Point (2). In Step 5 we prove that the Lipschitz constant (resp. Total Variation) of $\hat u$, $\hat v_k$ (resp. $\hat\sigma_k$) are uniformly bounded, Point (3).

Step 1. First we define the metric space Γ and the contraction $\mathcal{T}$. Consider the Banach space X defined in (5.12) and its subset

$$\begin{aligned}\Gamma := \Big\{ & \gamma = (u, v_1, \dots, v_N, \sigma_1, \dots, \sigma_N) \in X \text{ such that} \\ & u, v_1, \dots, v_N \text{ are Lipschitz and } \sigma_1, \dots, \sigma_N \text{ are } BV, \\ & \mathrm{Lip}(u),\ \mathrm{Lip}(v_k),\ \text{e.Tot.Var.}(\sigma_k) \leq C \text{ for all } k = 1, \dots N, \\ & u(0) = v_1(0) = \dots = v_N(0) = 0, \\ & \text{for every } k \in \{1, \dots, N\} \text{ and for every } z \in [0, M_k],\ v_k(z) \begin{cases} \geq 0 & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) = +1, \\ = 0 & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) = 0, \\ \leq 0 & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) = -1, \end{cases} \\ & |u(z)| \leq \delta \text{ for every } z \in [0, M] \\ & |v_k(z)| \leq \delta \text{ for every } k \in \{1, \dots, N\} \text{ and for every } z \in [0, M_k], \\ & |\sigma_k(z) - \lambda_k(0)| \leq \delta \text{ for every } k \in \{1, \dots, N\} \text{ and for } \mathcal{L}^1\text{-a.e. } z \in [0, M_k] \Big\}, \end{aligned} \tag{5.18}$$

where $C, \delta > 0$ will be chosen later. Clearly Γ is a closed subset of the Banach space X and thus it is a complete metric space. Denote by $\tilde{D}$ the distance induced on Γ by the norm $\|\cdot\|_\dagger$ of X. Notice that if $\delta \ll 1$, then r^γ, λ^γ are well defined for every $\gamma \in \Gamma$.

Consider now the transformation

$$\mathcal{T} : \Gamma \to X, \qquad \gamma = (u, v_1, \dots, v_N, \sigma_1, \dots, \sigma_N) \mapsto \mathcal{T}(\gamma) = \tilde{\gamma} = (\tilde{u}, \tilde{v}_1, \dots, \tilde{v}_N, \tilde{\sigma}_1, \dots, \tilde{\sigma}_N), \tag{5.19}$$

defined by the formula

$$\begin{cases} \tilde{u}(z) := \int_{(0,z]} V_\sharp\big(\rho r^\gamma \mathcal{L}^1|_{(L_0, L_N]}\big)(d\zeta), \\ \tilde{v}_k(z) := \begin{cases} f_k^\gamma(z) - \mathrm{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} f_k^\gamma(z), & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) \geq 0, \\ 0 & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) = 0, \\ \mathrm{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} f_k^\gamma(z) - f_k^\gamma(z), & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) < 0, \end{cases} & k = 1, \dots, N, \\ \tilde{\sigma}_k(z) := \frac{d}{dz} \mathrm{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} f_k^\gamma(z), & k = 1, \dots, N, \end{cases}$$

where r^γ, f_k^γ are defined in (5.13), (5.14) respectively. Notice that, by Proposition 5.11, for every $z \in (0, M_k]$, $\hat{\mathtt{x}}_k^{-1}\big(\hat{\mathtt{x}}_k(z)\big)$ is an interval in $\mathbb{R}$, while, by Lemma 5.18, the map f_k^γ is Lipschitz; therefore, by the property of the convex functions, $\tilde{\sigma}_k$ is well defined as a map in L^1. Recall also that if $I = \{z\}$ is made by a single point, then $\mathrm{conv}_I\, g(z) = g(z)$ and $d\, \mathrm{conv}_I\, g(z)/dz = dg(z)/dz$.

Step 2. We prove now that if $\delta, M \ll 1$ and $C \gg 1$, then, $\mathcal{T}(\Gamma) \subseteq \Gamma$. Let us start with the Lipschitz continuity of $\tilde{u}$. For $z_1 < z_2$, $z_1, z_2 \in [0, M_k]$, it holds

$$\begin{aligned}
|\tilde{u}(z_2) - \tilde{u}(z_1)| &= \left| \int_{(z_1,z_2]} V_\sharp\big(\rho r^\gamma \mathcal{L}^1|_{(L_0,L_N]}\big)(d\zeta) \right| \\
&= \left| \int_{V^{-1}((z_1,z_2])} \rho(w) r^\gamma(w) dw \right| \\
&\le \sup_k \|\tilde{r}_k\|_\infty \int_{V^{-1}((z_1,z_2])} \bar{\rho}(w) dw \\
&\le \sup_k \|\tilde{r}_k\|_\infty \int_{V^{-1}((z_1,z_2])} \bar{\rho}(w) dw \\
\text{(by Lemma 5.6)} \quad &\le \sup_k \|\tilde{r}_k\|_\infty (z_2 - z_1) \\
&\le C|z_2 - z_1|,
\end{aligned}$$

for C sufficiently large.
We prove now the Lipschitz continuity of $\tilde{v}_k$, $k = 1, \dots, N$. Take $z_1 < z_2$. Assume first that $\hat{\mathtt{x}}_k(z_1) = \hat{\mathtt{x}}_k(z_2) =: \bar{x}$. If $\mathcal{S}_k(x) = 0$ there is nothing to prove. If $\mathcal{S}_k(x) \neq 0$, then

$$\begin{aligned}
\big|\tilde{v}_k(z_2) - \tilde{v}_k(z_1)\big| &= \left| \Big(f_k^\gamma(z_2) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\bar{x})} f_k^\gamma(z_2) \Big) - \Big(f_k^\gamma(z_1) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\bar{x})} f_k^\gamma(z_1) \Big) \right| \\
&\le \Big| f_k^\gamma(z_2) - f_k^\gamma(z_1) \Big| + \left| \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\bar{x})} f_k^\gamma(z_2) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\bar{x})} f_k^\gamma(z_1) \right| \qquad (5.20) \\
\text{(by Lemma (5.18))} \quad &\le 2\big\|\tilde{\lambda}_k\big\|_\infty |z_2 - z_1| \\
&\le C|z_2 - z_1|,
\end{aligned}$$

if $C \gg 1$. Assume now that $\hat{\mathtt{x}}_k(z_1) < \hat{\mathtt{x}}_k(z_2)$. In this case for every $\eta > 0$ one can always find $z' \in \hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z_1))$ close enough to $\sup \hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z_1))$ and $z'' \in \hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z_2))$ close enough to $\inf \hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z_2)))$ such that

$$z_1 \le z' \le z'' \le z_2$$

and

$$\big|\tilde{v}_k(z')\big| \le \eta, \qquad \big|\tilde{v}_k(z'')\big| \le \eta.$$

We have

$$\begin{aligned}
\big|\tilde{v}_k(z_2) - \tilde{v}_k(z_1)\big| &\le \big|\tilde{v}_k(z_2) - \tilde{v}_k(z'')\big| + \big|\tilde{v}_k(z'') - \tilde{v}_k(z')\big| + \big|\tilde{v}_k(z') - \tilde{v}_k(z_1)\big| \\
&\le \big|\tilde{v}_k(z_2) - \tilde{v}_k(z'')\big| + \big|\tilde{v}_k(z'')\big| + \big|\tilde{v}_k(z')\big| + \big|\tilde{v}_k(z') - \tilde{v}_k(z_1)\big| \\
\text{(by (5.20))} \quad &\le C\big(z_2 - z''\big) + \big|\tilde{v}_k(z'')\big| + \big|\tilde{v}_k(z')\big| + C\big(z' - z_1\big) \\
&\le C\big(z_2 - z_1\big) + 2\eta,
\end{aligned}$$

and thus, by the arbitrarity of η,

$$\big|\tilde{v}_k(z_2) - \tilde{v}_k(z_1)\big| \le C\big|z_2 - z_1\big|.$$

Let us prove now that for every $k = 1, \dots, N$, $\tilde{\sigma}_k$ is BV and e.Tot.Var.$(\tilde{\sigma}_k) \le C$. Fix $k = 1, \dots, N$. By Lemma 5.18, we already know that

$$\text{e.Tot.Var.}\left(\frac{df_k^\gamma}{dz}\right) \le \mathcal{O}(1)\Big[\Big(\mathrm{Lip}(u) + \mathrm{Lip}(v_k)\Big)M + \text{e.Tot.Var.}(\sigma_k)\Big].$$

Let $g_k : [0, M_k] \to \mathbb{R}$ be a good representative for the total variation of $\frac{df_k^\gamma}{dz}$, i.e. $g_k = \frac{df_k^\gamma}{dz}$ for $\mathcal{L}^1$-a.e. $z \in [0, M_k]$ and

$$\text{e.Tot.Var.}\left(\frac{df_k^\gamma}{dz}; (0, M_k)\right) = \text{e.Tot.Var.}\big(g_k; (0, M_k)\big) = \text{p.Tot.Var.}\big(g_k; (0, M_k)\big). \tag{5.21}$$

Let

$$E_k := \left\{z \in [0, M_k] \;\middle|\; \frac{df_k^\gamma}{dz} \text{ does not exist}\right\} \cup \left\{z \in [0, M_k] \;\middle|\; g_k(z) \neq \frac{df_k^\gamma}{dz}(z)\right\}. \tag{5.22}$$

Clearly $\mathcal{L}^1(E) = 0$. Let us now prove that

$$\text{p.Tot.Var.}\big(\tilde{\sigma}_k; (0, M_k) \setminus E\big) \leq \text{p.Tot.Var.}\left(\frac{df_k^\gamma}{dz}; (0, M_k) \setminus E\right).$$

Take any $z_1 < \cdots < z_P$, $z_p \in (0, M_k) \setminus E$ for every $p = 1, \dots, P$ and define $Z := \{z_1, \dots, z_p\}$. Since:

- we want to estimate $\text{p.Tot.Var.}\big(\tilde{\sigma}_k; (0, M_k) \setminus E\big)$
- on each fixed x, $\tilde{\sigma}_k|_{\hat{\mathtt{x}}_k^{-1}(x)}$ is computed as the convex envelope of f_k^γ on $\hat{\mathtt{x}}_k^{-1}(x)$,

for computing the total variation we can assume w.l.o.g. that for every $x \in \mathbb{R}$ and for every $\sigma^* \in \mathbb{R}$,

$$\text{card}\left(Z \cap \hat{\mathtt{x}}_k^{-1}(z) \cap \tilde{\sigma}_k^{-1}(\sigma^*)\right) \leq 1, \qquad \text{card}\left(Z \cap \hat{\mathtt{x}}_k^{-1}(z)\right) \leq 2. \tag{5.23}$$

Now observe that for every $p = 1, \dots, P$, it is possible to find two points $z_p' \leq z_p \leq z_p''$ in $(0, M_k) \setminus E$ such that

$$z_1' \leq z_1'' \leq z_2' \leq z_2'' \leq \cdots \leq z_P' \leq z_p'' \tag{5.24}$$

and

$$\frac{df_k^\gamma}{dz}(z_p') \geq \tilde{\sigma}_k(z_p) \geq \frac{df_k^\gamma}{dz}(z_p''). \tag{5.25}$$

Indeed, if

- either $\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z_p))$ contains only z_p,
- or $\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z_p))$ is an interval with Lebesgue measure greater than zero, but z_p is a rarefaction point for $\text{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z_p))} f_k^\gamma$ i.e. the convex envelope coincides with f_k^γ,

then set $z_p' = z_p'' := z_p$. Otherwise, if $\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z_p))$ is an interval with Lebesgue measure greater than zero and z_p is a shock point for $\text{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(\tilde{z}_p))} f_k^\gamma$, then the existence of the points z_p', z_p'' is given by Proposition 1.12 and the assumption (5.23) implies (5.24).
Therefore

$$\tilde{\sigma}_k(z_p) - \tilde{\sigma}_k(z_{p-1}) \leq \left|\frac{d\hat{f}_k^\gamma}{dz}(z_p') - \frac{d\hat{f}_k^\gamma}{dz}(z_{p-1}'')\right|,$$

$$\begin{aligned}
&\tilde{\sigma}_k(z_{p-1}) - \tilde{\sigma}_k(z_p) \\
&\quad \leq \frac{d\hat{f}_k^\gamma}{dz}(z_{p-1}') - \frac{d\hat{f}_k^\gamma}{dz}(z_p'') \\
&\quad \leq \left|\frac{d\hat{f}_k^\gamma}{dz}(z_{p-1}') - \frac{d\hat{f}_k^\gamma}{dz}(z_{p-1}'')\right| + \left|\frac{d\hat{f}_k^\gamma}{dz}(z_{p-1}'') - \frac{d\hat{f}_k^\gamma}{dz}(z_p')\right| + \left|\frac{d\hat{f}_k^\gamma}{dz}(z_p') - \frac{d\hat{f}_k^\gamma}{dz}(z_p'')\right|
\end{aligned}$$

and thus

$$\begin{aligned}
&|\tilde{\sigma}_k(z_p) - \tilde{\sigma}_k(z_{p-1})| \\
&\quad \leq \left|\frac{d\hat{f}_k^\gamma}{dz}(z_{p-1}') - \frac{d\hat{f}_k^\gamma}{dz}(z_{p-1}'')\right| + \left|\frac{d\hat{f}_k^\gamma}{dz}(z_{p-1}'') - \frac{d\hat{f}_k^\gamma}{dz}(z_p')\right| + \left|\frac{d\hat{f}_k^\gamma}{dz}(z_p') - \frac{d\hat{f}_k^\gamma}{dz}(z_p'')\right|.
\end{aligned}$$

Hence we conclude

$$\sum_{p=2}^{P} |\tilde{\sigma}_k(z_p) - \tilde{\sigma}_k(z_{p-1})| \leq 3 \cdot \text{p.Tot.Var.}\Big(\frac{df_k^\gamma}{dz}; (0, M_k) \setminus E\Big).$$

and thus taking the supremum over all finite sequences $(z_p)_p$ in $(0, M_k) \setminus E$ we get

$$\text{p.Tot.Var.}\Big(\tilde{\sigma}_k; (0, M_k) \setminus E\Big) \leq 3 \cdot \text{p.Tot.Var.}\Big(\frac{df_k^\gamma}{dz}; (0, M_k) \setminus E\Big).$$

To prove that $\tilde{\sigma}_k$ is BV and to estimate its total variation, let us observe that, by Corollary 1.38,

$$\begin{aligned}
\text{e.Tot.Var.}(\tilde{\sigma}_k; (0, M_k)) &\leq \text{p.Tot.Var.}(\sigma_k; (0, M_k) \setminus E) \\
&\leq 3 \cdot \text{p.Tot.Var.}\Big(\frac{df_k^\gamma}{dz}; (0, M_k) \setminus E\Big) \\
&= 3 \cdot \text{p.Tot.Var.}\Big(g; (0, M_k) \setminus E\Big) \\
&\leq 3 \cdot \text{p.Tot.Var.}\Big(g; (0, M_k)\Big) \\
\text{(by (5.21))} \quad &= 3 \cdot \text{e.Tot.Var.}\Big(\frac{df_k^\gamma}{dz}; (0, M_k) \setminus E\Big) \\
\text{(by Lemma 5.18)} \quad &\leq \mathcal{O}(1)\bigg[\Big(\text{Lip}(u) + \text{Lip}(v_k)\Big)M + \bigg\|\frac{\partial \tilde{\lambda}_k}{\partial \sigma_k}\bigg\|_\infty \text{e.Tot.Var.}(\sigma_k)\bigg] \\
&\leq \mathcal{O}(1)\bigg[\Big(\text{Lip}(u) + \text{Lip}(v_k)\Big)M + \delta\text{e.Tot.Var.}(\sigma_k)\bigg] \\
&\leq \mathcal{O}(1)\Big[2CM + C\delta\Big] \\
&\leq C
\end{aligned}$$

for $M \ll 1$, $\delta \ll 1$.
We thus get that $\tilde{u}$, $\tilde{v}_k$, $k = 1, \dots, N$, are Lipschitz, σ_k, $k = 1, \dots, N$, are BV and

$$\text{Lip}(\tilde{u}),\ \text{Lip}(\tilde{v}_k),\ \text{e.Tot.Var.}(\tilde{\sigma}_k) \leq C \text{ for all } k = 1, \dots, N.$$

Clearly $\tilde{u}(0) = 0$. Now notice that $\tilde{v}_k$ is not defined in $z = 0$ by formula (5.19), since the domain of the map $\hat{\mathtt{x}}_k$ is $(0, M_k]$ and thus it does not contain $z = 0$. We already know that $\tilde{v}_k$ is Lipschitz continuous and thus it is enough to prove that $\lim_{z \to 0} \tilde{v}_k(z) = 0$. This is achieved as follows. Assume first that there exists some $\bar{x} \in \mathbb{R}$ such that $\inf \hat{\mathtt{x}}_k^{-1}(\bar{x}) = 0$. In this case for every z sufficiently close to 0 we have, by definition,

$$|\tilde{v}_k(z)| \leq \Big| f_k^\gamma(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} f_k^\gamma(z)\Big|$$

and thus $|v_k(z)| \to 0$ as $z \to 0$. Now assume that such a $\bar{x}$ does not exist. Hence, for every sequence $(z_p)_{p \in \mathbb{N}}$ such that $z_p \to 0$, it holds

$$\mathcal{L}^1\Big(\hat{\mathtt{x}_k}^{-1}(\hat{\mathtt{x}}_k(z_p))\Big) \to 0 \text{ as } p \to \infty.$$

Observing that

$$|\tilde{v}_k(z_p)| \leq \big\|\tilde{\lambda}_k\big\|_\infty \mathcal{L}^1\Big(\hat{\mathtt{x}_k}^{-1}(\hat{\mathtt{x}}_k(z_p))\Big)$$

we get $\tilde{v}_k(z_p) \to 0$ as $p \to \infty$ and thus $\lim_{z \to 0} \tilde{v}_k(z) = 0$.

The fact that for every $k \in \{1, \dots, N\}$ and for every $z \in [0, M_k]$,

$$\tilde{v}_k(z) \begin{cases} \geq 0 & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) = +1, \\ = 0 & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) = 0, \\ \leq 0 & \text{if } \mathcal{S}_k(\hat{\mathtt{x}}_k(z)) = -1, \end{cases}$$

is an easy consequence of the definition of $\tilde{v}_k$.

Let us now prove that $\tilde{u}$ remains uniformly close to zero and for every k, $\tilde{v}_k, \tilde{\sigma}_k$ remains uniformly close to 0 and $\lambda_k(0)$ respectively, whenever $M \ll 1$. We have

$$|\tilde{u}(z)| = |\tilde{u}(z) - \tilde{u}(0)| \leq C|z| \leq CM \leq \delta,$$

if $M \ll 1$. Similarly, for every k,

$$|\tilde{v}_k(z)| = |\tilde{v}_k(z) - \tilde{v}_k(0)| \leq C|z| \leq CM_k \leq CM \leq \delta,$$

if $M \ll 1$. Now observe that for every k and for $\mathcal{L}^1$-a.e. $z \in [0, M_k]$ it holds

$$\begin{aligned} \left| \frac{df_k^\gamma}{dz}(z) - \lambda_k(0) \right| &= \Big| \tilde{\lambda}_k\big(u(\omega_k(z)), v_k(z), \sigma_k(z)\big) - \tilde{\lambda}_k\big(u(0), 0, \sigma_k(z)\big) \Big| \\ &\leq \mathcal{O}(1)\Big(\big| u(\omega_k(z)) - u(0) \big| + \big| v_k(z) - v_k(0) \big| \Big) \\ &\leq \mathcal{O}(1) C \Big(\omega_k(z) + z \Big) \\ &\leq \mathcal{O}(1) CM \\ &\leq \delta \end{aligned}$$

if $M \ll 1$. Now use a Proposition 1.11 to get for $\mathcal{L}^1$-a.e. $z \in [0, M_k]$

$$\|\sigma_k - \lambda_k(0)\|_\infty \leq \left\| \frac{df_k^\gamma}{dz}(z) - \lambda_k(0) \right\|_\infty \leq \delta$$

if $M \ll 1$. We have thus proved that if $\delta, M \ll 1$ and $C \gg 1$, then $\mathcal{T}(\Gamma) \subseteq \Gamma$.
Step 3. We prove now that $\mathcal{T} : \Gamma \to \Gamma$ is a contraction, if $M, \delta \ll 1$. Let

$$\gamma = (u, v_1, \dots, v_N, \sigma_1, \dots, \sigma_N) \in \Gamma, \quad \gamma' = (u', v_1', \dots, v_N', \sigma_1', \dots, \sigma_N') \in \Gamma.$$

Define

$$\tilde{\gamma} := \mathcal{T}(\gamma) = \big(\tilde{u}, \tilde{v}_1, \dots, \tilde{v}_N, \tilde{\sigma}_1, \dots, \tilde{\sigma}_N\big) \in \Gamma$$

and

$$\tilde{\gamma}' := \mathcal{T}(\gamma') = \big(\tilde{u}', \tilde{v}_1', \dots, \tilde{v}_N', \tilde{\sigma}_1', \dots, \tilde{\sigma}_N'\big) \in \Gamma.$$

It holds

$$
\begin{aligned}
\big|\tilde u(z) - \tilde u'(z)\big| &= \bigg| \int_{(0,z]} V_\sharp\Big(\rho r^\gamma \mathcal{L}^1\Big)(d\zeta) - \int_{(0,z]} V_\sharp\Big(\rho r^{\gamma'} \mathcal{L}^1\Big)(d\zeta)\bigg| \\
&= \int_{V^{-1}((0,z])} \big|\rho(w)\big|\big|r^\gamma(w) - r^{\gamma'}(w)\big| dw \\
&\leq \sum_{k=1}^N \int_{V^{-1}((0,z])\cap(L_{k-1},L_k]} \bar\rho(w) \Big| \tilde r_k\big(u(V(w)), v_k(V_k(w)), \sigma_k(V_k(w))\big) \\
&\qquad\qquad - \tilde r_k\big(u'(V(w)), v_k'(V_k(w)), \sigma_k'(V_k(w))\big)\Big| dw \\
&\leq \sum_{k=1}^N \int_{(L_{k-1},L_k]} \bar\rho(w) \Big| \tilde r_k\big(u(V(w)), v_k(V_k(w)), \sigma_k(V_k(w))\big) \\
&\qquad\qquad - \tilde r_k\big(u'(V(w)), v_k'(V_k(w)), \sigma_k'(V_k(w))\big)\Big| dw \\
\text{(by (2.5))}\quad &\leq \mathcal{O}(1) \sum_{k=1}^N \int_{(L_{k-1},L_k]} \bar\rho(w) \Big(|u(V(w)) - u'(V(w))| + |v_k(V_k(w)) - v_k'(V_k(w))| \\
&\qquad\qquad + \delta |\sigma_k(V_k(w)) - \sigma_k'(V_k(w))| \Big) \\
&\leq \mathcal{O}(1) \bigg(M\|u - u'\|_\infty + \sum_{k=1}^N M_k \|v_k - v_k'\|_\infty \\
&\qquad\qquad + \delta \sum_{k=1}^N \int_{(L_{k-1},L_k]} \bar\rho(w) |\sigma_k(V_k(w)) - \sigma_k'(V_k(w))| dw \bigg)
\end{aligned}
$$

(making the change of variable $\zeta = V_k(w)$)

$$
\begin{aligned}
&= \mathcal{O}(1) \bigg(M\|u - u'\|_\infty + M \sum_{k=1}^N \|v_k - v_k'\|_\infty \\
&\qquad\qquad + \delta \sum_{k=1}^N \int_{(L_{k-1},L_k]} |\sigma_k(\zeta) - \sigma_k'(\zeta)| d\zeta \bigg) \\
&\leq \mathcal{O}(1) \max\{M, \delta\} \tilde D(\gamma, \gamma') \\
&\leq \frac{1}{2} \cdot \frac{1}{2N+1} \tilde D(\gamma, \gamma'),
\end{aligned}
$$

if $M, \delta \ll 1$.

For every $k = 1, \dots, N$, arguing as in the previous computation, we get

$$
\begin{aligned}
\bigg\| \frac{df_k^\gamma}{dz} - \frac{df_k^{\gamma'}}{dz} \bigg\|_1 &= \int_0^{M_k} \Big| \tilde\lambda_k\big(u(\omega_k(\zeta)), v_k(\zeta), \sigma_k(\zeta)\big) - \tilde\lambda_k\big(u'(\omega_k(\zeta)), v_k'(\zeta), \sigma_k'(\zeta)\big) \Big| d\zeta \\
&= \mathcal{O}(1) \bigg(M\|u - u'\|_\infty + M \sum_{k=1}^N \|v_k - v_k'\|_\infty + \delta \sum_{k=1}^N \|\sigma_k - \sigma_k'\|_1 \bigg) \\
&\leq \mathcal{O}(1) \max\{M, \delta\} \tilde D(\gamma, \gamma') \\
&\leq \frac{1}{4} \cdot \frac{1}{2N+1} \tilde D(\gamma, \gamma'),
\end{aligned}
$$

and thus

$$\|f_k^\gamma - f_k^{\gamma'}\|_\infty \leq \left\| \frac{df_k^\gamma}{dz} - \frac{df_k^{\gamma'}}{dz} \right\|_1 \leq \frac{1}{4} \cdot \frac{1}{2N+1} \tilde{D}(\gamma, \gamma'). \tag{5.26}$$

Hence for every $z \in [0, M_k]$ we have

$$\begin{aligned} |\tilde{v}_k(z) - \tilde{v}_k'(z)| &\leq \left| \left(f_k^\gamma(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} f_k^\gamma(z) \right) - \left(f_k^{\gamma'}(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} f_k^{\gamma'}(z) \right) \right| \\ \text{(by Proposition 1.11)} \quad &\leq 2 \|f_k^\gamma - f_k^{\gamma'}\|_\infty \\ \text{(by (5.26))} \quad &\leq \frac{1}{2} \cdot \frac{1}{2N+1} \tilde{D}(\gamma, \gamma'). \end{aligned}$$

Similarly, using again Proposition 1.11, we have

$$\|\tilde{\sigma}_k - \tilde{\sigma}_k'\|_1 = \int_{(0, M_k]} |\tilde{\sigma}_k(z) - \tilde{\sigma}_k'(z)| dz \leq \left\| \frac{df_k^\gamma}{dz} - \frac{df_k^{\gamma'}}{dz} \right\|_1 \leq \frac{1}{4} \cdot \frac{1}{2N+1} \tilde{D}(\gamma, \gamma').$$

Hence

$$\tilde{D}(\tilde{\gamma}, \tilde{\gamma}') = \|\tilde{u} - \tilde{u}'\|_\infty + \sum_{k=1}^N \|\tilde{v}_k - \tilde{v}_k'\|_\infty + \sum_{k=1}^N \|\tilde{\sigma}_k - \tilde{\sigma}_k'\|_1 \leq \frac{1}{2} \tilde{D}(\gamma, \gamma'),$$

thus proving that $\mathcal{T}$ is a contraction with contractive constant equal to $\frac{1}{2}$. We have thus proved Point (1) in the statement of the proposition, i.e. the existence of a curve

$$\hat{\gamma} := (\hat{u}, \hat{v}_1, \dots, \hat{v}_N, \hat{\sigma}_1, \dots, \hat{\sigma}_N),$$

with $\hat{u}$, $\hat{v}_k$ Lipschitz and σ_k in BV, $k = 1, \dots, N$, which solves the system (5.17).
Step 4. If $\hat{\gamma}$ and $\hat{\gamma}'$ are as in Point (2) in the statement of the proposition, then we can always find C large enough in the definition (5.18) of Γ such that

$$\mathrm{Lip}(\hat{u}), \mathrm{Lip}(\hat{u}'),\ \mathrm{Lip}(v_k), \mathrm{Lip}(v_k'),\ \mathrm{Tot.Var.}(\sigma_k), \mathrm{Tot.Var.}(\sigma_k') \leq C.$$

If both $\hat{\gamma}$ and $\hat{\gamma}'$ solves the fixed point problem (5.17), then they are fixed points of the contraction $\mathcal{T}$ and thus they must coincide.
Step 5. Finally observe that, by the computations in Steps 1,2,3, we can always choose C in the definition (5.18) of Γ depending only of f, thus getting Point (3) in the statement of the proposition. □

5.2. Definition of Lagrangian representation and statement of the main theorem

Using the tools we introduced in the previous section, we can now go the the heart of this chapter, with the definition of *Lagrangian representation* of the solution of the Cauchy problem (5.1) and the main theorem of this chapter, namely the existence of a Lagrangian representation.

We first introduce the notion of k-th characteristic speed of a point $(t, x) \in [0, \infty) \times \mathbb{R}$.

DEFINITION 5.20. For every family $k = 1, \dots, N$, the *k-th characteristic speed on the (t, x)-plane* is the function

$$\lambda_k : [0, \infty) \times \mathbb{R} \to \mathbb{R}, \qquad \lambda_k(t, x) := \begin{cases} \frac{1}{s_k} \int_0^{s_k} \sigma_k(\tau) d\tau & \text{if } s_k \neq 0, \\ \sigma_k(0) & \text{if } s_k = 0, \end{cases} \tag{5.27}$$

where we are assuming that the Riemann problem $(u(t, x-), u(t, x+))$ is solved by the collection of curves $\{\gamma_k,\ k = 1, \dots, N\}$, $\gamma_k = (u_k, v_k, \sigma_k) : \mathbf{I}(s_k) \to \mathcal{D}_k \subseteq \mathbb{R}^{N+2}$.

We have already used the notation λ_k to denote the k-th eigenvalue of the matrix $A(u) = DF(u)$. In that case, λ_k was a function of u, $\lambda_k = \lambda_k(u)$, while in this case λ_k is a function of (t,x). Therefore, no confusion should occur in the following.
Notice that, as pointed out in Section 2.1, we assume that, if $s_k = 0$, then γ_k is made by one single point, $\gamma_k(0) = \big(u_k(0), 0, \lambda_k(u_k(0))\big)$, where

$$u_k(0) = \begin{cases} u_{k-1}(s_{k-1}) & \text{if } k \geq 2, \\ u(t, x-) & \text{if } k = 1. \end{cases}$$

We can now give the most important definition of this paper, the definition of *Lagrangian representation.*

Definition 5.21. Let $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}^N$ be a solution of the Cauchy problem (5.1). A *Lagrangian representation* for u up to a fixed time $T > 0$ is a $(N+4)$-tuple $(L_0, \dots, L_N, \mathtt{x}, \rho, \bar{\rho})$, where

$$\begin{aligned} &L_0 \leq \cdots \leq L_N, \text{ and } (L_{k-1}, L_k] \text{ is called the } \textit{set of k-waves}, \\ &\mathtt{x} : [0,T] \times (L_0, L_N] \to \mathbb{R} \text{ is the } \textit{position function}, \\ &\rho : [0,T] \times (L_0, L_N] \to [-1,1] \text{ is the } \textit{density function}, \\ &\bar{\rho} : [0,T] \times (L_0, L_N] \to [-1,1] \text{ is the } \textit{absolute density function}, \end{aligned}$$

and, for every time $t \in [0,T]$ up to a countable set, the following properties hold:

(a) the $(N+4)$-tuple $\mathcal{E}(t) := \big(L_0, \dots, L_N, \mathtt{x}(t), \rho(t), \bar{\rho}(t)\big)$ is an enumeration of waves;

(b) the distributional derivative of $u(t,\cdot)$ w.r.t. x satisfies

$$D_x u(t) = \mathtt{x}(t)_\sharp \Big(\rho(t) r^{\hat{\gamma}(t)} \mathcal{L}^1|_{(L_0, L_N]} \Big),$$

where $\hat{\gamma}(t)$ is the solution of the fixed point problem (5.17) given by Proposition 5.19 associated to the enumeration of waves $\mathcal{E}(t)$ and $r^{\hat{\gamma}}$ is defined in (5.13);

(c) for every $x \in \mathbb{R}$, there exists at most one family $k \in \{1, \dots, N\}$ such that

$$\int_{\mathtt{x}^{-1}(\bar{x}) \cap (L_{k-1}, L_k]} \bar{\rho}(t,w) \neq 0$$

and for every wave $w \in \mathtt{x}^{-1}(t)(x) \cap (L_{k-1}, L_k]$,

$$\hat{\sigma}_k(t, V_k(t,w)) = \lambda_k(t,x).$$

In addition,

(d) extending on the whole $\mathbb{R}^2$ the maps $\rho, \bar{\rho}$ to zero outside the set $[0,T] \times (L_0, L_N]$, the distributions $D_t\rho, D_t\bar{\rho}$ are finite Radon measure on $\mathbb{R}^2$;

(e) for every fixed $w \in (L_0, L_N]$ the map $t \mapsto \mathtt{x}(t,w)$ is 1-Lipschitz and moreover

$$\frac{\partial \mathtt{x}}{\partial t}(t,w) = \lambda_k(t, \mathtt{x}(t,w)), \text{ for } \bar{\rho}(\cdot, w)\mathcal{L}^1\text{-a.e. time } t \in [0,T]. \tag{5.28}$$

Let us add some remarks about the definition of Lagrangian representation. First of all observe that Points (a)-(c) of Definition 5.21 describe *static* properties of the objects $(L_0, \dots, L_N, \mathtt{x}, \rho, \bar{\rho})$, i.e. properties at a fixed time $t \in [0,T]$, while Points (d)-(e) describe *dynamic* properties, i.e. they describe the behavior of the objects $(L_0, \dots, L_N, \mathtt{x}, \rho, \bar{\rho})$ when time goes on. In particular

- Point (a) guarantees that at a.e. time $t \in [0,T]$ we can use Proposition 5.19 to construct the fixed point curve

$$\hat{\gamma}(t) = \big(\hat{u}(t), \hat{v}_1(t), \dots, \hat{v}_N(t), \hat{\sigma}_1(t), \dots, \hat{\sigma}_N(t)\big)$$

and thus also $r^{\hat{\gamma}(t)}, \lambda^{\hat{\gamma}(t)}, f_k^{\hat{\gamma}(t)}$;
- Point (b)-(c) says that, given the maps $\mathtt{x}(t), \rho(t), \bar{\rho}(t)$, we can use $\hat{\gamma}(t)$ to recover the solution $u(t)$; moreover the maps $\hat{\sigma}_k(t)$, $k = 1, \dots, N$, give the characteristic speed of $u(t,x)$ at any continuity point of $u(t)$ and the speed of the shock $(u(t,x-), u(t,x+))$ at any jump point of $u(t)$;
- Point (d) guarantees that the mass of a.e. waves $w \in (L_0, L_N]$ is a BV function in time, i.e. almost no wave can be created and canceled too many times;
- Point (e) says that for a.e. wave $w \in (L_0, L_N]$ and a.e. time $t \in [0,T]$ (when the wave has non-zero density) the trajectory of the wave w is, in some sense, a characteristic curve, because its derivative at a.e. time coincides with the characteristic speed.

In the next section we will prove the existence of a Lagrangian representation for the solution $u(t,x)$ to the Cauchy problem (5.1). In particular, we will prove the following theorem.

THEOREM C. *Let $u(t) := S_t\bar{u}$ be the vanishing viscosity solution of the Cauchy problem (5.1) with initial datum $\bar{u}$. Let $T > 0$ be a fixed time. Then there exists a Lagrangian representation of u up to the time T, which moreover satisfies the following condition: up to countable many times, for every $x \in \mathbb{R}$*

$$x \text{ is a continuity point for } u(t,\cdot) \iff \int_{\mathtt{x}(t)^{-1}(x)} \rho(t,w)dw = 0. \tag{5.29}$$

We give now a sketch of the proof of Theorem C, which will be the topic of all the next sections.

SKETCH OF THE PROOF. Fix $T > 0$. Let $u(t,\cdot) = S_t\bar{u}$ be the solution of the Cauchy problem (5.1). We already know, by Theorem B proved in Chapter 4, that u can be obtained as limit of the Glimm approximations u^ε. Let us now divide the proof in several steps.

Step 1. In Section 3.3, Theorem 3.25, we have shown that for every Glimm approximate solution u^ε it is possible to construct a wave tracing

$$\mathcal{E}^\varepsilon = \big(L_0^\varepsilon, \dots, L_N^\varepsilon, \mathtt{x}^\varepsilon, \rho^\varepsilon\big)$$

for u^ε up to time T and the related map $\bar{\sigma}^\varepsilon(t,w)$ defined on $[0,T] \times (L_0, L_N]$ as in (3.31). Set also

$$\bar{\rho}^\varepsilon(t,w) := |\rho^\varepsilon(t,w)|.$$

Notice that in Section 3.3 the dependence of $\mathcal{E}^\varepsilon$ on ε was not explicitly noted. Here, however, we write this dependence explicitly because we are now interested in passing to the limit as $\varepsilon \to 0$.
It is not difficult to see that the numbers $L_0^\varepsilon, \dots, L_N^\varepsilon$ can be chosen in such a way that they do not depend on ε. Indeed, since u^ε is identically zero out of a compact set in $[0,T] \times \mathbb{R}$, we can always add some "artificial wave" to $(L_{k-1}^\varepsilon, L_k^\varepsilon]$ located, at time $t = 0$, out of this compact set, moving with constant speed equal to 1 and with density equal to zero for every time $t \in [0,T]$. Define thus the set of k-waves (independent of ε)

$$\mathcal{W}_k := (L_{k-1}, L_k].$$

Step 2. Passing to the limit (in some appropriate topology) the maps $\mathtt{x}^\varepsilon$, ρ^ε, $\bar{\rho}^\varepsilon$, we will show in Section 5.5, Propositions 5.38 and 5.39, that it is possible to construct three maps

$\mathtt{x} : [0,T] \times (L_0, L_N] \to \mathbb{R}$ is the *position function*,

$\rho : [0,T] \times (L_0, L_N] \to [-1,1]$ is the *density function*,

$\bar{\rho} : [0,T] \times (L_0, L_N] \to [-1,1]$ is the *absolute density function*, which, together with the numbers $L_0 \leq \cdots L_N$, will be the candidate Lagrangian representation. We have thus to prove that they satisfy Properties (a)-(e) above and the additional property (5.29).

Step 3. Property (a) and (d), which depend only on $\mathtt{x}, \rho, \bar{\rho}$ and not on the related objects whose construction we presented in Section 5.1, will be an easy consequence of the correspondent properties in the approximations $\mathtt{x}^\varepsilon$, ρ^ε, $\bar{\rho}^\varepsilon$. This will be shown again in Propositions 5.38 and 5.39.

Step 4. Finally the proof of the properties (b), (c), (e) and the additional property (5.29), which involve also all the objects whose construction is presented in Section 5.1, will be performed in Section 5.7. □

Let us now make a summary of the work we are going to do in the next sections. As we pointed out in the sketch of the proof of Theorem C, the proofs of Properties (b), (c), (e) and of the additional property (5.29) involve also the objects whose construction is presented in Section 5.1. More precisely, observe that, for every fixed time $t \in [0,T]$, the $(N+4)$-tuple

$$\big((L_0, \dots, L_N, \mathtt{x}^\varepsilon(t,\cdot), \rho^\varepsilon(t,\cdot), \bar{\rho}^\varepsilon(t,\cdot)\big),$$

is an enumeration of waves in the sense of Definition 5.1. Therefore, according to the analysis in Section 5.1, we can construct,

- the sign $\mathcal{S}_k^\varepsilon(t)$ of points $x \in \mathbb{R}$ (see (5.2));
- the order relation $\Re^\varepsilon(t)$ on $(L_0, L_N]$ (see (5.3));
- the numbers $M_k^\varepsilon(t), M^\varepsilon(t) \in \mathbb{R}$ (see (5.4));
- the maps $V_k^\varepsilon(t), V^\varepsilon(t), \omega_k^\varepsilon(t)$ (see (5.5) and (5.7));
- the maps $\hat{\mathtt{x}}_k^\varepsilon(t), \hat{\mathtt{x}}^\varepsilon(t)$ (see (5.8));
- the curve $\hat{\gamma}^\varepsilon(t) := \big(\hat{u}^\varepsilon(t), \hat{v}_1^\varepsilon(t), \dots, \hat{v}_N^\varepsilon(t), \sigma_1^\varepsilon(t), \dots, \sigma_N^\varepsilon(t)\big)$, (see Proposition 5.19).

Notice that the construction of all the above objects is done at every fixed time $t \in [0,T]$ and thus they all depend on t.

By Propositions 5.38 and 5.39, also the $(N+4)$-tuple of the limit objects, at every fixed time $t \in [0,T]$,

$$\big((L_0, \dots, L_N, \mathtt{x}(t,\cdot), \rho(t,\cdot), \bar{\rho}(t,\cdot)\big)$$

is an enumeration of waves in the sense of Definition 5.1. Therefore, similarly to what we have just done for the approximations, we can contruct, as in Section 5.1:

- the sign $\mathcal{S}_k(t)$ of points $x \in \mathbb{R}$ (see (5.2));
- the order relation $\Re(t)$ on $(L_0, L_N]$ (see (5.3));
- the numbers $M_k(t), M(t) \in \mathbb{R}$ (see (5.4));
- the maps $V_k(t), V^(t), \omega_k(t)$ (see (5.5) and (5.7));
- the maps $\hat{\mathtt{x}}_k(t), \hat{\mathtt{x}}(t)$ (see (5.8));
- the curve $\hat{\gamma}(t) := \big(\hat{u}(t), \hat{v}_1(t), \dots, \hat{v}_N(t), \sigma_1(t), \dots, \sigma_N(t)\big)$, (see Proposition 5.19).

As before, notice that the construction of all the above objects is done at every fixed time $t \in [0,T]$ and thus they all depend on t.

The technique we will follow to prove Properties (b), (c), (e) and the additional property (5.29) in Section 5.7 will be based on the fact that the objects

$$\Re,\ M_k,\ M,\ V_k,\ V,\ \hat{\mathtt{x}}_k,\ \hat{\mathtt{x}},\ \hat{u},\ \hat{v}_k,\ \hat{\sigma}_k$$

constructed with the techniques of Section 5.1 starting from $\mathtt{x}, \rho, \bar{\rho}$ are the limits of the corresponding objects

$$\Re^\varepsilon,\ M_k^\varepsilon,\ M^\varepsilon,\ V_k^\varepsilon,\ V^\varepsilon,\ \hat{\mathtt{x}}_k^\varepsilon,\ \hat{\mathtt{x}}^\varepsilon,\ \hat{u}^\varepsilon,\ \hat{v}_k^\varepsilon,\ \hat{\sigma}_k^\varepsilon$$

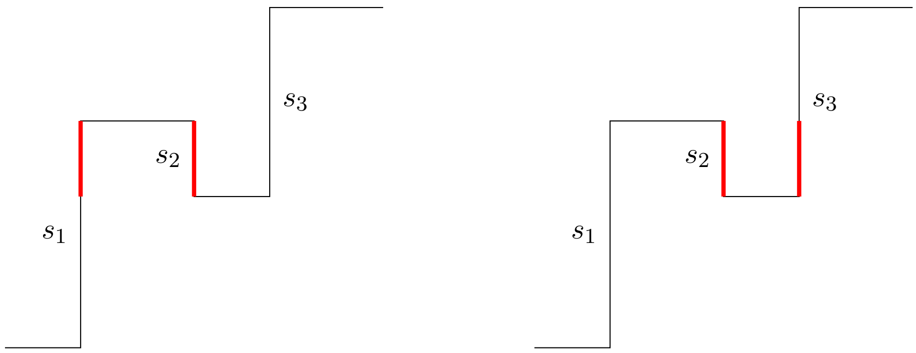

FIGURE 8. Non uniqueness of the Lagrangian representation. The wavefronts s_1, s_2, s_3 in the exact solution collide in the same point $(\bar{t}, \bar{x})$. In the approximations (where only binary collisions take place) two situations can occur: if the first interaction is between s_1 and s_2, then the canceled waves (in red in the left picture) belong to s_1 and s_2; on the other side, if the first interaction is between s_2 and s_3, then the canceled waves (in red in the right picture) belong to s_2 and s_3.

constructed with the techniques of Section 5.1 starting from $\mathtt{x}^\varepsilon, \rho^\varepsilon, \bar{\rho}^\varepsilon$ (up to subsequence and in the appropriate topologies). This will be the aim of Section 5.5, where we will use a careful analysis of some estimates which allow to control the interactions among many Riemann problems (Section 5.3) and a careful analysis of each Glimm approximate solution u^ε (Section 5.4).

We conclude this section with the following two observations. A detailed analysis about the topics of these two remarks will appear in [BM15a].

REMARK 5.22. Theorem C provides the existence of at least one Lagrangian representation. In general, however, many Lagrangian representations are possible for the same solution of the Cauchy problem (5.1). This is the case, for instance, when three wavefronts (one positive wavefront s_1, one negative wavefront s_2 and another one positive wavefront s_3) collide in the same point $(\bar{t}, \bar{x})$; in this situation, indeed, different approximations (in which only binary collisions take place) can lead to different Lagrangian representation, according to the portion of waves which are canceled in the collision. See Figure 8.

However, a sort of *stability* of the Lagrangian representation can be recovered, in the sense that it is possible to prove what follows, using exactly the same computations as the ones developed in the next sections. Assume that (u^n) is a sequence of *exact* (not approximate) solutions to the Cauchy problem (5.1) and u is another solution to the same Cauchy problem (with different initial data). If $\mathcal{E}^n = (L_0, \dots, L_N, \mathtt{x}^n, \rho^n, \bar{\rho}^n)$ is a Lagrangian representation for u^n up to time T, then $\mathcal{E}^n$ converges to a Lagrangian representation $\mathcal{E} = (L_0, \dots, L_N, \mathtt{x}, \rho, \bar{\rho})$ of u, in the sense that $\mathtt{x}^n(t) \to \mathtt{x}(t)$ in L^1, $\rho^n(t) \to \rho(t)$ weakly* in L^∞, $\bar{\rho}^n(t) \to \bar{\rho}(t)$ weakly* in L^∞ and all the objects

$$\Re,\ M_k,\ M,\ V_k,\ V,\ \hat{\mathtt{x}}_k,\ \hat{\mathtt{x}},\ \hat{u},\ \hat{v}_k,\ \hat{\sigma}_k$$

constructed with the techniques of Section 5.1 starting from $\mathtt{x}, \rho, \bar{\rho}$ can be recovered as the limits of the corresponding objects

$$\Re^\varepsilon,\ M_k^\varepsilon,\ M^\varepsilon,\ V_k^\varepsilon,\ V^\varepsilon,\ \hat{\mathtt{x}}_k^\varepsilon,\ \hat{\mathtt{x}}^\varepsilon,\ \hat{u}^\varepsilon,\ \hat{v}_k^\varepsilon,\ \hat{\sigma}_k^\varepsilon$$

constructed with the techniques of Section 5.1 starting from $\mathtt{x}^\varepsilon, \rho^\varepsilon, \bar{\rho}^\varepsilon$ (up to subsequence and in the appropriate topologies).

Remark 5.23. The presence of two maps $\rho, \bar{\rho}$ in the definition of Lagrangian representation is due to the lower semi-continuity of the weak* convergence, as we will see in Proposition 5.39. In general, in fact, passing to the limit ρ^ε and $\bar{\rho}^\varepsilon$ we will get maps ρ, $\bar{\rho}$ such that $|\rho|$ is strictly less than $\bar{\rho}$ in a set of positive Lebesgue measure.

However, we think that it is possible to construct a Lagrangian representation in which $\bar{\rho}(t) = |\rho(t)|$ for a.e. time $t \in [0,T]$. The idea of the proof of this fact is as follows. First we can force the condition $\bar{\rho}(\bar{t}) = |\rho(\bar{t})|$ at a fixed given time $\bar{t}$. Now, for fixed n and for any $i = 0, 1, \dots, n-1$ we construct a Lagrangian representation on the time interval $[iT/n, (i+1)T/n]$ with the property that, at time iT/n, $\bar{\rho}$ and $|\rho|$ coincide. Then the Lagrangian representations on each time interval $[iT/n, (i+1)T/n]$ can be concatenated, thus obtaining a Lagrangian representation on the time interval $[0,T]$ with the property that, at each time iT/n, $\bar{\rho}$ and $|\rho|$ coincide. Passing to the limit as $n \to \infty$ we get the conclusion.

5.3. Local interaction estimates among many Riemann problems

Before starting the proof of Theorem C, we need first to prove some "local" interaction estimate, in the same spirit of the analysis in Sections 3.2 and 4.4.
In particular the situation we have in mind to study is the following. Fix $\varepsilon > 0$, $t \in [0,T]$, $x, x' \in \mathbb{R}$ such that $u^\varepsilon(t,\cdot)$ is continuous at x, x'. The state $u^\varepsilon(t,x)$ is connected to the state $u^\varepsilon(t,x')$ through a sequence of NP exact curves $\{\gamma_k^p\}_k^p$, $k = 1, \dots, N$, $p = 1, \dots, P$, where P is the number of the discontinuity points $x_1 < \cdots x_P$ of $u^\varepsilon(t,\cdot)$ between x and x'. For any p, the curves $\{\gamma_k^p\}_k$ connect $u^\varepsilon(t, x^p-)$ with $u^\varepsilon(t, x^p+)$.
The results we are going to prove now state, roughly speaking, that if the speeds σ_k^p are close to a constant σ^*, then the solution of the Riemann problem $(u^\varepsilon(t,x), u^\varepsilon(t,x'))$ is close to a shock or contact discontinuity traveling with speed σ^*. This is fairly easy to see in the scalar case where the reduced flux coincides with the flux F of the Cauchy problem (5.1). In the system case the analysis requires more effort and it is more technical.

We now forget about this motivating example and we start studying a more abstract situation. We first consider a situation when all the curves belong to the same family. Let thus k be a fixed family. Let us consider a collection of P exact curves of the k-th family, $\gamma_k^1, \dots, \gamma_k^P$, with length $s_k^1, \dots, s_k^P$ respectively. The components of γ_k^p are $\gamma_k^p = (u_k^p, v_k^p, \sigma_k^p)$ and the reduced flux associated to γ_k^p is f_k^p. We assume that they are consecutive and they satisfy the assumption ($\star$) (see page 35). Set also

$$a_k^p := \sum_{q=1}^p s_k^p.$$

and

$$I_k^p := a_k^{p-1} + \mathbf{I}(s_k^p).$$

Let $\gamma_k = (u_k, v_k, \sigma_k)$ be the exact curve of the k-th family with length a_k^P starting in $u_k^1(0)$ and let f_k be the associated reduced flux. Assume that γ_k and f_k are defined on $I_k := \mathbf{I}(a_k^P)$.

All the next results are obtained always assuming that the total variation is small enough, i.e. $\sum_p |s_k^p| \ll 1$ (depending only on F). The first lemma we prove estimates the distance between the first and the last curve, γ_k^1 and γ_k^P respectively, restricted to the part of their domain which belong to $I_k^1 \cap I_k^P$ but has empty intersection with I_k^p for $p = 2, \dots, P-1$.

LEMMA 5.24. *Let $\sigma^* \in \mathbb{R}$ be any constant. Assume that $s_k^1 \cdot s_k^P \leq 0$. Then for every τ in the closure of $(I_k^1 \cap I_k^P) \setminus \bigcup_{p=2}^{P-1} I_k^p$, it holds*

$$\left.\begin{array}{r}\big|u_k^P(\tau) - u_k^1(\tau)\big| \\ \big|v_k^1(\tau)\big| \\ \big|v_k^P(\tau)\big|\end{array}\right\} \leq \mathcal{O}(1) \sum_{p=1}^{P} \big\|\sigma_k^p - \sigma^*\big\|_{L^1(I_k^p)}.$$

PROOF. Set $s := \sum_p |s_k^p|$. The set $(I_k^1 \cap I_k^P) \setminus \bigcup_{p=2}^{P-1} I_k^p$ is not empty only if one of the following four cases occurs:

(1) $a_k^P \leq a_k^0 \leq \min\{a_k^1, \dots, a_k^{P-1}\}$;

(2) $a_k^0 \leq a_k^P \leq \min\{a_k^1, \dots, a_k^{P-1}\}$;

(3) $\max\{a_k^1, \dots, a_k^{P-1}\} \leq a_k^0 \leq a_k^P$;

(4) $\max\{a_k^1, \dots, a_k^{P-1}\} \leq a_k^P \leq a_k^0$.

We prove the lemma only in the case (1). All the other cases can be treated similarly. Let us thus assume that

$$a_k^P \leq a_k^0 \leq \min\{a_k^1, \dots, a_k^{P-1}\} \tag{5.30}$$

and in this case the closure of $(I_k^1 \cap I_k^P) \setminus \bigcup_{p=2}^{P-1} I_k^p$ is $[a_k^0, \min\{a_k^1, \dots, a_k^{P-1}]$. Let us also assume, for simplicity, that $s_k^p \neq 0$ for every p. Define first the following sets of indices:

$$\mathcal{I}^+ := \big\{p = 1, \dots, P \;\big|\; s_k^p > 0\big\}, \qquad \mathcal{I}^- := \big\{p = 1, \dots, P \;\big|\; s_k^p < 0\big\}.$$

For every $p \in \mathcal{I}^+$ and for every $\tau \in I_k^p$ set

$$\alpha(p, \tau) := \min\big\{q \in \mathcal{I}^- \;\big|\; q > p \text{ and } \tau \in I_k^q\big\}.$$

and for every $q \in \mathcal{I}^-$ and for every $\tau \in I_k^q$ set

$$\beta(q, \tau) := \max\big\{p \in \mathcal{I}^+ \;\big|\; p < q \text{ and } \tau \in I_k^p\big\}.$$

Observe that, thanks to (5.30),

(a) the definition is well posed;

(b) for every $p \in \mathcal{I}^+$ and $q \in \mathcal{I}^-$

$$\big\{\tau \in I_k^p \;\big|\; \alpha(p, \tau) = q\big\} = \big\{\tau \in I_k^q \big|\; \beta(q, \tau) = p\big\}; \tag{5.31}$$

(c) for fixed $p \in \mathcal{I}^+$ and for fixed $\tau \in I_k^p$, setting $q := \alpha(p, \tau)$, we have that for every $p' \in \mathcal{I}^+$, $p \leq p' \leq q$,

$$I_k^{p'} \cap [\tau, +\infty) = \bigcup_{\substack{q' \in \mathcal{I}^- \\ p \leq q' \leq q}} \big\{\varsigma \in I_k^{p'} \;\big|\; \alpha(p', \varsigma) = q'\big\} \cap [\tau, +\infty), \tag{5.32a}$$

with $p < q' \leq q$, and conversely for every $q' \in \mathcal{I}^-$, $p \leq q' \leq q$,

$$I_k^{q'} \cap [\tau, +\infty) = \bigcup_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \big\{\varsigma \in I_k^{q'} \;\big|\; \beta(q', \varsigma) = p'\big\} \cap [\tau, +\infty), \tag{5.32b}$$

with $p \leq p' < q$.

Fix now any $p \in \mathcal{I}^+$ and take any $\tau \in I_k^p$. Let $q := \alpha(p,\tau)$. It holds

$$\begin{aligned} u_k^q(\tau) - u_k^p(\tau) &= \int_{a^{q-1}}^{\tau} \tilde{r}_k(\gamma_k^q) d\varsigma + \sum_{p'=p+1}^{q} \int_{a^{p'-1}}^{a^{p'}} \tilde{r}_k(\gamma_k^{p'}) d\varsigma + \int_{\tau}^{a_k^p} \tilde{r}_k(\gamma_k^p) d\varsigma \\ &= \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{I_k^{p'} \cap [\tau, +\infty)} \tilde{r}_k(\gamma_k^{p'}) d\varsigma - \sum_{\substack{q' \in \mathcal{I}^- \\ p \leq q' \leq q}} \int_{I_k^{q'} \cap [\tau, +\infty)} \tilde{r}_k(\gamma_k^{q'}) d\varsigma. \end{aligned} \tag{5.33}$$

Now observe that, by (5.32)

$$\begin{aligned} \sum_{\substack{q' \in \mathcal{I}^- \\ p \leq q' \leq q}} \int_{I_k^{q'} \cap [\tau, +\infty)} \tilde{r}_k(\gamma_k^{q'}) d\varsigma &= \sum_{\substack{q' \in \mathcal{I}^- \\ p \leq q' \leq q}} \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{\{\varsigma \in I_k^{q'} \mid \beta(q',\varsigma) = p'\} \cap [\tau, +\infty)} \tilde{r}_k(\gamma_k^{q'}) d\varsigma \\ \text{(by (5.31))} \quad &= \sum_{\substack{q' \in \mathcal{I}^- \\ p \leq q' \leq q}} \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{\{\varsigma \in I_k^{p'} \mid \alpha(p',\varsigma) = q'\} \cap [\tau, +\infty)} \tilde{r}_k\big(\gamma_k^{\alpha(p',\varsigma)}\big) d\varsigma \\ &= \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \sum_{\substack{q' \in \mathcal{I}^- \\ p \leq q' \leq q}} \int_{\{\varsigma \in I_k^{p'} \mid \alpha(p',\varsigma) = q'\} \cap [\tau, +\infty)} \tilde{r}_k\big(\gamma_k^{\alpha(p',\varsigma)}\big) d\varsigma \\ &= \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{I_k^{p'} \cap [\tau, +\infty)} \tilde{r}_k\big(\gamma_k^{\alpha(p',\varsigma)}\big) d\varsigma. \end{aligned}$$

Therefore we can continue the chain of equalities in (5.33) as follows:

$$\begin{aligned} u_k^q(\tau) - u_k^p(\tau) &= \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{I_k^{p'} \cap [\tau, +\infty)} \tilde{r}_k(\gamma_k^{p'}) d\varsigma - \sum_{\substack{q' \in \mathcal{I}^- \\ p \leq q' \leq q}} \int_{I_k^{q'} \cap [\tau, +\infty)} \tilde{r}_k(\gamma_k^{q'}) d\varsigma \\ &= \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{I_k^{p'} \cap [\tau, +\infty)} \tilde{r}_k(\gamma_k^{p'}) d\varsigma - \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{I_k^{p'} \cap [\tau, +\infty)} \tilde{r}_k\big(\gamma_k^{\alpha(p',\varsigma)}\big) d\varsigma \\ &= \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{I_k^{p'} \cap [\tau, +\infty)} \Big[\tilde{r}_k(\gamma_k^{p'}) - \tilde{r}_k\big(\gamma_k^{\alpha(p',\varsigma)}\big)\Big] d\varsigma. \end{aligned}$$

Hence

$$\begin{aligned} |u_k^q(\tau) - u_k^p(\tau)| &\leq \sum_{\substack{p' \in \mathcal{I}^+ \\ p \leq p' \leq q}} \int_{I_k^{p'} \cap [\tau, +\infty)} \Big|\tilde{r}_k(\gamma_k^{p'}) - \tilde{r}_k\big(\gamma_k^{\alpha(p',\varsigma)}\big)\Big| d\varsigma \\ &\leq \mathcal{O}(1)|s| \Bigg\{ \sup_{\substack{p' \in \mathcal{I}^+ \\ \varsigma \in I_k^{p'}}} \big|u_k^{p'}(\varsigma) - u_k^{\alpha(p',\varsigma)}(\varsigma)\big| + \sup_{\substack{p' \in \mathcal{I}^+ \\ \varsigma \in I_k^{p'}}} \big|v_k^{p'}(\varsigma) - v_k^{\alpha(p',\varsigma)}(\varsigma)\big| \\ &\qquad + \sum_{p' \in \mathcal{I}^+} \int_{I_k^{p'}} \big|\sigma_k^{p'}(\varsigma) - \sigma_k^{\alpha(p',\varsigma)}(\varsigma)\big| d\varsigma \Bigg\} \end{aligned}$$

and thus

$$\begin{aligned}\sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|u_k^{\alpha(p,\tau)}(\tau) - u_k^p(\tau)\right| \leq \mathcal{O}(1)|s|\Bigg\{ &\sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|u_k^p(\tau) - u_k^{\alpha(p,\tau)}(\tau)\right| + \sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|v_k^p(\tau) - v_k^{\alpha(p,\tau)}(\tau)\right| \\ &+ \sum_{p\in\mathcal{I}^+} \int_{I^p} \left|\sigma_k^p(\tau) - \sigma_k^{\alpha(p,\tau)}(\tau)\right| d\tau \Bigg\}.\end{aligned} \tag{5.34}$$

A completely similar argument, with $\tilde{\lambda}_k$ instead of $\tilde{r}_k$, shows that

$$\begin{aligned}\sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|f_k^{\alpha(p,\tau)}(\tau) - f_k^p(\tau)\right| \leq \mathcal{O}(1)|s|\Bigg\{ &\sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|u_k^p(\tau) - u_k^{\alpha(p,\tau)}(\tau)\right| + \sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|v_k^p(\tau) - v_k^{\alpha(p,\tau)}(\tau)\right| \\ &+ \sum_{p\in\mathcal{I}^+} \int_{I^p} \left|\sigma_k^p(\tau) - \sigma_k^{\alpha(p,\tau)}(\tau)\right| d\tau \Bigg\}.\end{aligned} \tag{5.35}$$

Now fix any $p \in \mathcal{I}^+$ and any $\tau \in I_k^p$, using the same argument as before, we can write

$$\begin{aligned}|v_k^p(\tau)| &= f_k^p(\tau) - \operatorname{conv}_{I_k^p}(\tau) \\ &= \left(f_k^p(\tau) - f_k^{\alpha(p,\tau)}(\tau)\right) + \left(f_k^{\alpha(p,\tau)}(\tau) - \operatorname{conc}_{I^{\alpha(p,\tau)}} f_k^{\alpha(p,\tau)}(\tau)\right) \\ &\quad + \left(\operatorname{conc}_{I^{\alpha(p,\tau)}} f_k^{\alpha(p,\tau)}(\tau) - \operatorname{conv}_{I_k^p} f_k^p(\tau)\right) \\ &\leq \sup_{\substack{p'\in\mathcal{I}^+\\ \varsigma\in I_k^{p'}}} \left|f_k^{\alpha(p',\tau)}(\varsigma) - f_k^{p'}(\varsigma)\right| + \sum_{p'\in\mathcal{I}^+} \int_{I_k^{p'}} \left|\sigma_k^{p'}(\varsigma) - \sigma_k^{\alpha(p',\tau)}(\varsigma)\right| d\varsigma\end{aligned} \tag{5.36}$$

and similarly for every $q \in \mathcal{I}^-$ and any $\tau \in I_k^q$

$$|v_k^q(\tau)| \leq \sup_{\substack{p'\in\mathcal{I}^+\\ \varsigma\in I_k^{p'}}} \left|f_k^{\alpha(p',\tau)}(\varsigma) - f_k^{p'}(\varsigma)\right| + \sum_{p'\in\mathcal{I}^+} \int_{I_k^{p'}} \left|\sigma_k^{p'}(\varsigma) - \sigma_k^{\alpha(p',\tau)}(\varsigma)\right| d\varsigma. \tag{5.37}$$

Hence for any $p \in \mathcal{I}^+$ and for any $\tau \in I^p$,

$$\begin{aligned}\sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} |v^p(\tau) - v^{\alpha(p,\tau)}| &\leq \sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} |v^p(\tau)| + \sup_{\substack{q\in\mathcal{I}^-\\ \tau\in I^q\cap[0,\infty)}} |v^q(\tau)| \\ &\leq 2\Bigg\{ \sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|f^{\alpha(p,\tau)}(\tau) - f^p(\tau)\right| + \sum_{p\in\mathcal{I}^+} \int_{I^p} \left|\sigma^p(\tau) - \sigma^{\alpha(p,\tau)}(\tau)\right| d\tau \Bigg\}.\end{aligned} \tag{5.38}$$

Now, using (5.34), (5.35), (5.38), we get that, if $|s| \ll 1$ (depending only on F), then

$$\sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|u_k^{\alpha(p,\tau)}(\tau) - u_k^p(\tau)\right| + \sup_{\substack{p\in\mathcal{I}^+\\ \tau\in I^p}} \left|v_k^{\alpha(p,\tau)}(\tau) - v_k^p(\tau)\right| \leq \mathcal{O}(1) \sum_{p\in\mathcal{I}^+} \int_{I^p} \left|\sigma_k^p(\tau) - \sigma_k^{\alpha(p,\tau)}(\tau)\right| d\tau \tag{5.39}$$

Observing that if $a_k^0 \leq \tau \leq \min\{a_k^1, \dots, a_k^{P-1}\}$, then $\alpha(1,\tau) = P$, from (5.39) we get

$$\left|u_k^P(\tau) - u_k^1(\tau)\right| \leq \mathcal{O}(1) \sum_{p=1}^{P} \left\|\sigma_k^p - \sigma^*\right\|_{L^1(I_k^p)}$$

for every $\tau \in \left[a_k^0, \min\{a_k^1, \dots, a_k^{P-1}\}\right]$.
Finally, from (5.36), (5.37), (5.35) and (5.39), it follows

$$\left|v_k^1(\tau)\right| \leq \mathcal{O}(1) \sum_{p=1}^{P} \left\|\sigma_k^p - \sigma^*\right\|_{L^1(I_k^p)}, \qquad \left|v_k^P(\tau)\right| \leq \mathcal{O}(1) \sum_{p=1}^{P} \left\|\sigma_k^p - \sigma^*\right\|_{L^1(I_k^p)}$$

for every $\tau \in \left[a_k^0, \min\{a_k^1, \dots, a_k^{P-1}\}\right]$. □

Lemma 5.25. *Assume that all the s_k^p have the same sign. Then for every constant $\sigma^* \in \mathbb{R}$*

$$D\Big(\gamma_k, \bigcup_{p=1}^{P} \gamma_k^p\Big) \leq \mathcal{O}(1) \sum_{p=1}^{P} \|\sigma_k^p - \sigma^*\|_{L^1(I_k^p)},$$

where D is the distance among curves introduced in Section 2.1.2.

Proof. Set for simplicity

$$\hat{\sigma} := D \operatorname*{conv}_{[a_k^0, a_k^P]} \bigcup_{p=1}^{P} f_k^p.$$

We know by Lemma 3.14 that

$$\begin{aligned} D\Big(\gamma, \bigcup_{p=1}^{P} \gamma_k^p\Big) \ &\leq \mathcal{O}(1) \int_{a_k^0}^{a_k^P} \left|\hat{\sigma}(\tau) - \Big(\bigcup_{p=1}^{P} \sigma_k^p\Big)(\tau)\right| d\tau \\ (\text{by Proposition 1.8}) \ &\leq \mathcal{O}(1) \sum_{p=1}^{P} \left\|\sigma_k^p - \sigma^*\right\|_1, \end{aligned}$$

thus concluding the proof of the lemma. □

The following Proposition related the curves $\{\gamma_k^p\}_p$ with the γ_k, in terms of the distance of the speeds σ_k^p from a fixed constant $\sigma^* \in \mathbb{R}$.

Proposition 5.26. *For every constant $\sigma^* \in \mathbb{R}$, the following estimates hold:*

$$\left|u_k^P(a_k^P) - u_k(a_k^P)\right| \leq \mathcal{O}(1) \sum_{p=1}^{P} \left\|\sigma_k^p - \sigma^*\right\|_{L^1(I_k^p)}; \tag{5.40a}$$

$$\|v_k^p\|_{L^\infty(I_k^p \,\cap\, \bigcup_{p' \neq p} I_k^{p'})} \leq \mathcal{O}(1) \sum_{q=1}^{P} \left\|\sigma_k^q - \sigma^*\right\|_{L^1(I_k^q)} \text{ for every } p = 1, \dots, P; \tag{5.40b}$$

$$\|v_k^p\|_{L^\infty(I_k^p \,\cap\, I_k)} \leq \mathcal{O}(1) \left[a_k^P + \sum_{q=1}^{P} \left\|\sigma_k^q - \sigma^*\right\|_{L^1(I_k^q)}\right] \text{ for every } p = 1, \dots, P; \tag{5.40c}$$

$$\left\|\sigma_k - \sigma^*\right\|_{L^1(\boldsymbol{I}(a_k^P))} \leq \mathcal{O}(1) \sum_{q=1}^{P} \left\|\sigma_k^q - \sigma^*\right\|_{L^1(I_k^q)}. \tag{5.40d}$$

PROOF. Let $\sigma^* \in \mathbb{R}$ be any constant. We assume that $a_k^P \geq 0$, the case $a_k^P \leq 0$ being analogous. Let us prove first (5.40b). Fix $p \in \{1, \dots, P\}$ and take $\tau \in I_k^p \cap \bigcup_{p' \neq p} I_k^{p'}$. Then either $\{p' < p \mid \tau \in I_k^{p'}\} \neq \emptyset$ or $\{p' > p \mid \tau \in I_k^{p'}\} \neq \emptyset$. Assume that $\{p' < p \mid \tau \in I_k^{p'}\} \neq \emptyset$, the other case being completely similar. Let $q := \max\{p' < p \mid \tau \in I_k^{p'}\}$. It is not difficult to see that we can apply Lemma 5.24 to the family of curves $\gamma_k^q, \gamma_k^{q+1}, \dots, \gamma_k^p$ to obtain

$$\left|v_k^p(\tau)\right| \leq \mathcal{O}(1) \sum_{p'=q}^{p} \|\sigma_k^{p'} - \sigma^*\|_{L^1(I_k^{p'})}.$$

Therefore

$$\|v_k^p\|_{L^\infty(I_k^p \,\cap\, \bigcup_{p' \neq p} I_k^{p'})} \leq \mathcal{O}(1) \sum_{q=1}^{P} \|\sigma_k^q - \sigma^*\|_{L^1(I_k^q)},$$

which is what we wanted to prove.

Let us prove now the other inequalities. Consider first the map $\beta : (a_k^0, a_k^P] \to \{1, \dots, P\}$ defined as

$$\beta(\tau) = \min\Big\{p \in \{1, \dots, P\} \;\Big|\; \tau \in I_k^p\Big\}.$$

Assume that

$$\beta\Big((a_k^0, a_k^P]\Big) = \{p_1, \dots, p_J\},$$

with $p_1 < \dots < p_J$ and set $K^j := \beta^{-1}(p_j)$. It not hard to see that β is increasing and thus each K^j is an interval of the form $K^j = (b^{j-1}, b^j]$. Set also $p_0 := 0$.
For every fixed $j = 1, \dots, J$, observe that the family of curves $\gamma_k^{p_{j-1}+1}, \dots, \gamma_k^{p_j}$ satisfies the hypothesis of Lemma 5.24. Therefore we can apply that lemma to obtain

$$\left|v_k^{p_j}(b^{j-1})\right| \leq \mathcal{O}(1) \sum_{p=p^{j-1}+1}^{p_j} \left\|\sigma_k^p - \sigma^*\right\| \tag{5.41}$$

and

$$\left|u_k^{p_j}(b^{j-1}) - u_k^{p_{j-1}}(b^{j-1})\right| \leq \mathcal{O}(1) \sum_{p=p^{j-1}+1}^{p_j} \left\|\sigma_k^p - \sigma^*\right\|. \tag{5.42}$$

Observe also that for $j = 1, \dots, J-1$,

$$v_k^{p_j}(b^j) = 0, \tag{5.43}$$

while a (iterated) application of Lemma 5.24 to the family of curves $\gamma_k^{p_J}, \dots, \gamma^P$ together with the triangular inequality yields

$$\left|v_k^{p_J}(b^J)\right| \leq \mathcal{O}(1) \sum_{p=p_J}^{P} \left\|\sigma_k^p - \sigma^*\right\| \tag{5.44}$$

and

$$\left|u_k^{p_J}(b^J) - u_k^P(a_k^P)\right| \leq \mathcal{O}(1) \sum_{p=p_J}^{P} \left\|\sigma_k^p - \sigma^*\right\|. \tag{5.45}$$

Let us now define, for every $j = 1, \dots, J$, the curves $\tilde{\gamma}^j = (\tilde{u}^j, \tilde{v}^j, \tilde{\sigma}^j)$ as

$$\tilde{\gamma}^j := \Gamma_k\Big(u_k^{p_j}(b^{j-1}), b^j - b^{j-1}\Big).$$

We compute now the distance between the curve $\tilde{\gamma}^j$ and the restriction $\gamma_k^{p_j}|_{[b^{j-1},b^j]}$, for every j. By Lemma 3.9, if $j = 1, \dots, J-1$, then

$$\begin{aligned} D\Big(\gamma_k^{p_j}|_{[b^{j-1},b^j]}, \tilde{\gamma}^j\Big) &\le 2\Big(\big|v_k^{p_j}(b^{j-1})\big| + \big|v_k^{p_j}(b^j)\big|\Big) \\ \text{(by (5.41), (5.43))} &\le \mathcal{O}(1) \sum_{p=p^{j-1}+1}^{p_j} \big\|\sigma_k^p - \sigma^*\big\|, \end{aligned} \tag{5.46}$$

while for $j = J$,

$$\begin{aligned} D\Big(\gamma_k^{p_J}|_{[b^{J-1},b^J]}, \tilde{\gamma}^J\Big) &\le 2\Big(\big|v_k^{p_J}(b^{J-1})\big| + \big|v_k^{p_J}(b^J)\big|\Big) \\ \text{(by (5.41), (5.44))} &\le \mathcal{O}(1) \sum_{p=p^{J-1}+1}^{P} \big\|\sigma_k^p - \sigma^*\big\|, \end{aligned} \tag{5.47}$$

As a consequence, we have that for every $j = 2, \dots, J$,

$$\begin{aligned} \big|\tilde{u}_k^j(b^{j-1}) - \tilde{u}_k^{j-1}(b^{j-1})\big| &\le \big|\tilde{u}_k^j(b^{j-1}) - u_k^{p_{j-1}}(b^{j-1})\big| + \big|u_k^{p_{j-1}}(b^{j-1}) - \tilde{u}_k^{j-1}(b^{j-1})\big| \\ &\le \big|u_k^{p_j}(b^{j-1}) - u_k^{p_{j-1}}(b^{j-1})\big| + \big|u_k^{p_{j-1}}(b^{j-1}) - \tilde{u}_k^{j-1}(b^{j-1})\big| \\ \text{(by (5.42) and (5.46))} &\le \mathcal{O}(1) \sum_{p=p^{j-1}+1}^{p_j} \big\|\sigma_k^p - \sigma^*\big\|. \end{aligned} \tag{5.48}$$

Define, by recursion, the curves $\hat{\gamma}^j = (\hat{u}^j, \hat{v}^j, \hat{\sigma}^j)$ as

$$\hat{\gamma}_k^1 := \Gamma_k\Big(u^L, b^1 - b^0\Big) = \tilde{\gamma}^1, \qquad \hat{\gamma}^j := \Gamma_k\Big(\hat{u}^{j-1}(b^{j-1}), b^j - b^{j-1}\Big) \text{ for } j = 2, \dots, J.$$

To compute the distance between $\hat{\gamma}^j$ and $\tilde{\gamma}^j$, we can apply Lemma 3.8 and (5.48), obtaining

$$D\big(\hat{\gamma}^j, \tilde{\gamma}^j\big) \le \mathcal{O}(1) \sum_{p=1}^{p_j} \big\|\sigma_k^p - \sigma^*\big\|. \tag{5.49}$$

Finally

$$\begin{aligned} D\Big(\gamma, \bigcup_{j=1}^J \gamma_k^{p_j}|_{[b^{j-1},b^j]}\Big) &\le D\Big(\gamma, \bigcup_{j=1}^J \hat{\gamma}^j\Big) + D\Big(\bigcup_{j=1}^J \hat{\gamma}^j, \bigcup_{j=1}^J \tilde{\gamma}^j\Big) + D\Big(\bigcup_{j=1}^J \tilde{\gamma}^j, \bigcup_{j=1}^J \gamma_k^{p_j}|_{[b^{j-1},b^j]}\Big) \\ &= D\Big(\gamma, \bigcup_{j=1}^J \hat{\gamma}^j\Big) + \sum_{j=1}^J D\big(\hat{\gamma}^j, \tilde{\gamma}^j\big) + \sum_{j=1}^J D\big(\tilde{\gamma}^j, \gamma_k^{p_j}|_{[b^{j-1},b^j]}\big) \\ &\qquad \text{(by Lemma 5.25, (5.46), (5.47) and (5.49))} \\ &\le \mathcal{O}(1) \sum_{p=1}^P \big\|\sigma_k^p - \sigma^*\big\|. \end{aligned} \tag{5.50}$$

Let us prove now (5.40c). If $\tau \in I_k^p \cap I_k$ and there exists $p' \neq p$ such that $\tau \in I_k^{p'}$, we can use (5.40b). If $\tau \in I_k^p \cap I_k$ and there is no $p' \neq p$ such that $\tau \in I_k^{p'}$, then $p = p_j$ for some j.

Therefore by (5.50), we have

$$\begin{aligned}|v_k^p(\tau)| &= |v_k^{p_j}(\tau)| \\ &\le |v_k^{p_j}(\tau) - v(\tau)| + |v(\tau)| \\ &\le \mathcal{O}(1)\bigg\{\sum_{p=1}^{P} \|\sigma_k^p - \sigma^*\| + |v(\tau)|\bigg\} \\ &\le \mathcal{O}(1)\bigg\{\sum_{p=1}^{P} \|\sigma_k^p - \sigma^*\| + a_k^P\bigg\},\end{aligned}$$

where the last inequality is a consequence of the fact that the length of the curve γ is a_k^P.

Let us prove now (5.40a). We have

$$\begin{aligned}|u_k^P(a_k^P) - u_k(a_k^P)| &\le |u_k^P(a_k^P) - u_k^{p_J}(a_k^P)| + |u_k^{p_J}(a_k^P) - u_k(a_k^P)| \\ \text{(by (5.45) and (5.50))} &\le \mathcal{O}(1)\sum_{p=1}^{P} \|\sigma_k^p - \sigma^*\|.\end{aligned}$$

Finally let us prove (5.40d). We have

$$\begin{aligned}\|\sigma_k - \sigma^*\|_{L^1(I)} &\le \Big\|\sigma_k - \bigcup_{j=1}^{J} \sigma_k^{p_j}\Big\|_{L^1(I)} + \sum_{j=1}^{J} \|\sigma_k^{p_j} - \sigma^*\|_1 \\ \text{(by (5.50))} &\le \mathcal{O}(1)\sum_{p=1}^{P} \|\sigma_k^p - \sigma^*\|,\end{aligned}$$

thus concluding the proof of the proposition. □

Let us consider now, as at the beginning of this section, the situation where there is more than one family. Let γ_k^p, $p = 1, \dots, P$, $k = 1, \dots, N$ be a collection of NP exact curves, with $P \in \mathbb{N} \setminus \{0\}$. Denote by $\gamma_k^p = (u_k^p, v_k^p, \sigma_k^p)$ the components of γ_k^p and by f_k^p the associated reduced fluxes. Assume that

(1) for every p, γ_k^p is an exact curve of the k-th family with length s_k^p;
(2) the starting point of the first curves γ_1^1 is a fixed state u^L;
(3) the curves $\{\gamma_k^p\}_k^p$ are consecutive w.r.t. the order

$$(p,k) \text{ precedes } (p',k') \iff p < p' \text{ or } p = p' \text{ and } k < k'.$$

Consider now another collection of NP curves $\{\tilde\gamma_k^p\}_k^p$, $p = 1, \dots, P$, $k = 1, \dots, N$. Denote by $\tilde\gamma_k^p = (\tilde u_k^p, \tilde v_k^p, \tilde\sigma_k^p)$ the components of $\tilde\gamma_k^p$ and by $\tilde f_k^p$ the associated reduced fluxes. Assume that

(1) for every p, $\tilde\gamma_k^p$ is an exact curve of the k-th family with length s_k^p;
(2) the starting point of the first curves $\tilde\gamma_1^1$ is u^L;
(3) the curves $\{\tilde\gamma_k^p\}_k^p$ are consecutive w.r.t. the order

$$(p,k) \text{ precedes } (p',k') \iff k < k' \text{ or } k = k' \text{ and } p < p'.$$

Observe that the curves $\{\tilde\gamma_k^p\}_k^p$ are obtained from the curves $\{\gamma_k^p\}_k^p$ after all the transversal interactions took place. As before, set

$$a_k^p := \sum_{q=1}^{p} s_k^p, \qquad I_k^p := a_k^{p-1} + \mathbf{I}(s_k^p).$$

We assume that for every fixed k, the collections of curves $\{\gamma_k^1, \dots \gamma_k^P\}$ and $\{\tilde{\gamma}_k^1, \dots, \tilde{\gamma}_k^P\}$ satisfy the assumption ($\star$).

Finally, for every k, define also the curve $\gamma_k = (u_k, v_k, \sigma_k)$ as the exact curve of the k-th family with length a_k^P, starting in $\tilde{u}_k^1(0)$. To explicitly stress the fact that the curve γ_k has been obtained starting from the collection of Np curves $\{\gamma_k^p\}_{k=1,\dots,N}^{p=1,\dots,P}$, we will write

$$\gamma_k = \mathcal{G}_k\Big(\{\gamma_k^p\}_{k=1,\dots,N}^{p=1,\dots,P}\Big). \tag{5.51}$$

The following proposition holds.

PROPOSITION 5.27. *For every constant $\sigma^* \in \mathbb{R}$, the following estimates hold:*

$$\big|u_1^1(0) - u_k(0)\big| \leq \mathcal{O}(1) \sum_{p=1}^P \sum_{h\neq k} |s_h^p|; \tag{5.52a}$$

$$\big|u_N^P(a_N^P) - u_k(a_k^P)\big| \leq \mathcal{O}(1) \sum_{p=1}^P \bigg[\sum_{h\neq k} |s_h^p| + \big\|\sigma_k^p - \sigma^*\big\|_{L^1(I_k^p)}\bigg]; \tag{5.52b}$$

$$\big\|v_k^p\big\|_{L^\infty(I_k^p \,\cap\, \bigcup_{p'\neq p} I_k^{p'})} \leq \mathcal{O}(1) \sum_{p=1}^P \bigg[\sum_{h\neq k} |s_h^p| + \big\|\sigma_k^p - \sigma^*\big\|_{L^1(I_k^p)}\bigg]; \tag{5.52c}$$

$$\big\|v_k^p\big\|_{L^\infty(I_k^p \,\cap\, I_k)} \leq \mathcal{O}(1) \bigg\{ a_k^P + \sum_{p=1}^P \bigg[\sum_{h\neq k} |s_h^p| + \big\|\sigma_k^p - \sigma^*\big\|_{L^1(I_k^p)}\bigg]\bigg\}; \tag{5.52d}$$

$$\big\|\sigma_k - \sigma^*\big\|_{L^1(\boldsymbol{I}(a_k^P))} \leq \mathcal{O}(1) \sum_{p=1}^P \bigg[\sum_{h\neq k} |s_h^p| + \big\|\sigma_k^p - \sigma^*\big\|_{L^1(I_k^p)}\bigg]. \tag{5.52e}$$

PROOF. First of all one passes from the curves $\{\gamma_k^p\}$ to the curve $\{\tilde{\gamma}_k^p\}$ using Corollary 3.13. Then, for every fixed family $k \in \{1, \dots, N\}$, one uses Proposition 5.26. $\square$

COROLLARY 5.28. *There exists a constant $C > 0$ depending only on f such that:*

- *if $a_k^P > 0$ then for every p and for every $z \in I_k^p$*

$$v_k^p(z) \geq -C \sum_{p=1}^P \bigg[\sum_{h\neq k} |s_h^p| + \big\|\sigma_k^p - \sigma^*\big\|_{L^1(I_k^p)}\bigg];$$

- *if $a_k^P < 0$ then for every p and for every $z \in I_k^p$*

$$v_k^p(z) \leq C \sum_{p=1}^P \bigg[\sum_{h\neq k} |s_h^p| + \big\|\sigma_k^p - \sigma^*\big\|_{L^1(I_k^p)}\bigg].$$

PROOF. The proof follows easily from (5.52c) and from the fact that if $z \in I_k^p \setminus \bigcup_{p'\neq p} I_k^{p'}$, then $\operatorname{sign}(s_k^p) = \operatorname{sign}(a_k^P)$. $\square$

5.4. Analysis of the approximate solutions

In this section we continue the analysis started in Chapter 3 on an approximate solution u^ε constructed by means of the Glimm scheme. In particular we will focus on those result which will be used in the next sections to conclude the proof of Theorem C. Let thus $\varepsilon > 0$ be fixed and let u^ε be the Glimm approximate solution with grid size ε.

5.4.1. Fixed point problem in the approximations. We show now that, for every time $t \in [0,T]$, the curve $\hat{\gamma}^\varepsilon(t)$, constructed with the technique of Section 5.1 starting from $\mathtt{x}^\varepsilon$ and ρ^ε, describes exactly all the discontinuities present in the Glimm approximation $u^\varepsilon(t,\cdot)$ as a function of x, at time t.

We have already observed at the end of Section 5.2 that, given the wave tracing

$$\mathcal{E}^\varepsilon = \big(L_0, \dots, L_N, \mathtt{x}^\varepsilon, \rho^\varepsilon\big)$$

for u^ε (up to time T) and the related map $\bar{\sigma}^\varepsilon(t,w)$ defined on $[0,T] \times (L_0, L_N]$ as in (3.31), for every fixed time $t \in [0,T]$, the $(N+4)$-tuple

$$\big((L_0, \dots, L_N, \mathtt{x}^\varepsilon(t,\cdot), \rho^\varepsilon(t,\cdot), \bar{\rho}^\varepsilon(t,\cdot)\big),$$

with $\bar{\rho}^\varepsilon(t,w) := |\rho^\varepsilon(t,w)|$ is an enumeration of waves in the sense of Definition 5.1. Therefore, according to the analysis in Section 5.1, we can construct,

- the sign $\mathcal{S}^\varepsilon_k(t)$ of points $x \in \mathbb{R}$ (see (5.2));
- the order relation $\Re^\varepsilon(t)$ on $(L_0, L_N]$ (see (5.3));
- the numbers $M^\varepsilon_k(t), M^\varepsilon(t) \in \mathbb{R}$ (see (5.4));
- the maps $V^\varepsilon_k(t), V^\varepsilon(t), \omega^\varepsilon_k(t)$ (see (5.5) and (5.7));
- the maps $\hat{\mathtt{x}}^\varepsilon_k(t), \hat{\mathtt{x}}^\varepsilon(t)$ (see (5.8));
- the curve

  $$\hat{\gamma}^\varepsilon(t) := \big(\hat{u}^\varepsilon(t), \hat{v}^\varepsilon_1(t), \dots, \hat{v}^\varepsilon_N(t), \sigma^\varepsilon_1(t), \dots, \sigma^\varepsilon_N(t)\big),$$

  and the functions $\hat{f}^\varepsilon_k(t) := f_k^{\hat{\gamma}^\varepsilon(t)}$, $k = 1, \dots, N$,

  $$f^\varepsilon_k(t) : [0, M^\varepsilon_k(t)] \to \mathbb{R}, \quad f^\varepsilon_k(t)(z) := \int_{(0,z]} \big(V^\varepsilon_k(t)\big)_\sharp \Big(\bar{\rho}^\varepsilon(t) \lambda^{\hat{\gamma}^\varepsilon(t)} \mathcal{L}^1|_{(L^\varepsilon_{k-1}, L^\varepsilon_k]}\Big)(d\zeta),$$

  such that $\hat{u}^\varepsilon, \hat{v}^\varepsilon_k$ are uniformly Lipschitz, $\hat{\sigma}^\varepsilon_k$ has uniformly bounded Total Variation for all $k = 1, \dots, N$ and they satisfy the fixed point system

$$\begin{cases} \hat{u}^\varepsilon(t)(z) := \int_{(0,z]} V^\varepsilon(t)_\sharp \big(\rho^\varepsilon(t) r^{\hat{\gamma}^\varepsilon(t)} \mathcal{L}^1|_{(L^\varepsilon_0, L^\varepsilon_N]}\big)(d\zeta), \\ \hat{v}^\varepsilon_k(t)(z) := \mathrm{sign}\Big(\mathcal{S}^\varepsilon_k(t)(\hat{\mathtt{x}}^\varepsilon_k(t)(z))\Big)\Big(f^\varepsilon_k(t)(z) - \mathrm{conv}_{(\hat{\mathtt{x}}^\varepsilon_k(t))^{-1}(\hat{\mathtt{x}}^\varepsilon_k(t)(z))} f^\varepsilon_k(t)(z)\Big), \quad k = 1, \dots, N, \\ \hat{\sigma}_k(z) := \frac{d}{dz} \mathrm{conv}_{(\hat{\mathtt{x}}^\varepsilon_k(t))^{-1}(\hat{\mathtt{x}}^\varepsilon_k(t)(z))} f^{\hat{\gamma}}_k(z), \quad k = 1, \dots, N. \end{cases}$$

  (see Proposition 5.19);

The following theorem explains the relation between $\hat{u}^\varepsilon(t,\cdot), \hat{v}^\varepsilon_k(t,\cdot), \hat{\sigma}^\varepsilon_k(t,\cdot)$ and the exact curves which solve the Riemann problem at each discontinuity point of $x \mapsto u^\varepsilon(t,x)$. Let us first introduce the following notation. Let $t \in [0,T]$ be a fixed time.
Let x_p, $p = 1, \dots, P$, be the discontinuity points of $x \mapsto u^\varepsilon(t,x)$. They are finite since we $u^\varepsilon(t,x)$ is equal to zero out of a compact set. Assume that, for all p, the Riemann problem $(u(t,x_p-), u(t,x_p+))$ is solved by

$$u(t, x_p+) = T^N_{s^p_N} \circ \cdots T^1_{s^p_1} u(t, x_p-)$$

and let $\gamma_k^p = (u_k^p, v_k^p, \sigma_k^p)$, $k = 1, \dots, N$, be the exact curves which solve $(u(t, x_p-), u(t, x_p+))$. Define also

$$u_0^p := u(t, x_p-), \quad u_k^p := T^k_{s_k^p} \circ \cdots T^1_{s_1^p}\, u(t, x_p-).$$

Set

$$I_k^p := \hat{\mathtt{x}}_k(t)^{-1}(x_p), \qquad J_k^p := \omega_k(t)(I_k^p)$$

(J_k^p is defined only if $I_k^p \neq 0$). We already now, by the general properties of an enumeration of waves (see Lemma 5.14) that

$$\omega_k : I_k^p \to J_k^p$$

is an affine map with slope equal to 1. Moreover, by Property (3) in the definition of wave tracing (Section 3.3) and by Lemma 5.12, for every $p = 1, \dots, P$,

$$|s_k| = \left| \int_{\mathtt{x}^\varepsilon(t)^{-1}(x_p)} \rho^\varepsilon(t, w) dw \right| = \int_{\mathtt{x}^\varepsilon(t)^{-1}(x_p)} \bar{\rho}^\varepsilon(t, w) dw = \mathcal{L}^1(I_k^p).$$

Therefore, we can assume that γ_k^p is defined on I_k^p if $s_k^p > 0$ or on $-I_k^p$ if $s_k^p < 0$, instead of $\mathbf{I}(s_k^p)$.

THEOREM 5.29. *For all $p = 1, \dots, P$ and for all $k = 1, \dots, N$ and for all $z \in I_k^p \neq 0$, it holds*

$$\big(u_k^p(z), v_k^p(z), \sigma_k^p(z)\big) = \begin{cases} \Big(\hat{u}^\varepsilon\big(t, \omega_k^\varepsilon(t, z)\big),\ \hat{v}_k^\varepsilon(t, z),\ \hat{\sigma}_k^\varepsilon(t, z)\Big) & \text{if } s_k^p > 0, \\ \Big(\hat{u}^\varepsilon\big(t, \omega_k^\varepsilon(t, -z)\big),\ \hat{v}_k^\varepsilon(t, -z),\ \hat{\sigma}_k^\varepsilon(t, -z)\Big) & \text{if } s_k^p < 0, \end{cases}$$

PROOF. Fix $p \in \{1, \dots, P\}$ and $k \in \{1, \dots N\}$. Since we are working at fixed time t, we omit the explicit dependence of the objects under consideration on t. Consider the curve $\tilde{\gamma}_k^p = (\tilde{u}_k^p, \tilde{v}_k^p, \tilde{\sigma}_k^p)$, whose domain is I_k^p if $s_k^p > 0$ or $-I_k^p$ if $s_k^p < 0$, defined as

$$\big(\tilde{u}_k^p(z), \tilde{v}_k^p(z), \tilde{\sigma}_k^p(z)\big) := \begin{cases} \Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(z)\big),\ \hat{v}_k^\varepsilon(z),\ \hat{\sigma}_k^\varepsilon(z)\Big) & \text{if } s_k^p > 0, \\ \Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(-z)\big),\ \hat{v}_k^\varepsilon(-z),\ \hat{\sigma}_k^\varepsilon(-z)\Big) & \text{if } s_k^p < 0, \end{cases}$$

and denote by $\tilde{f}_k^p$ the reduced flux associated to $\tilde{\gamma}_k^p$:

$$\tilde{f}_k^p(z) := \int_0^z \tilde{\lambda}_k\big(\tilde{\gamma}_k^p(\zeta)\big) d\zeta. \tag{5.53}$$

It is enough to prove that the curve $z \mapsto \big(\tilde{u}_k^p(z), \tilde{v}_k^p(z), \tilde{\sigma}_k^p(z)\big)$ solves the Riemann problem of lenght s_k^p and starting point u_{k-1}^p.

We first prove that the curve $z \mapsto (\tilde{u}_k^p, \tilde{v}_k^p, \tilde{\sigma}_k^p)$ solves a Riemann problem of length s_k^p and starting point

$$\int_{(0, \inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon} \mathcal{L}^1|_{(L_0, L_N]}\Big)(dz).$$

As in the proof of Lemma 5.18 we have

$$\begin{aligned}
\hat{u}^\varepsilon\big(\omega_k^\varepsilon(z)\big) &= \int_{(0,\omega_k^\varepsilon(z)]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1\Big)(d\zeta) \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{(\inf J_k^p,\omega_k^\varepsilon(z)]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{\omega_k((\inf I_k^p,z])} (\omega_k^\varepsilon)_\sharp (V_k^\varepsilon)_\sharp\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{(\inf I_k^p,z]} (V_k^\varepsilon)_\sharp\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{(V_k^\varepsilon)^{-1}((\inf I_k^p,z])} \rho^\varepsilon(w) r^{\hat{\gamma}^\varepsilon}(w)dw.
\end{aligned}$$

Notice now that, by definition of I_k^p, $|\rho^\varepsilon|\mathcal{L}^1$ a.e. wave in $(V_k^\varepsilon)^{-1}((\inf I_k^p, z])$ belongs to $(L_{k-1}, L_k]$ and has position x_p. Therefore, by the regularity properties of ρ^ε in w (Point (3) at page 52) we have that, on $(V_k^\varepsilon)^{-1}((\inf I_k^p, z])$, $\rho^\varepsilon(w) \geq 0$ if $s_k^p > 0$ or $\rho^\varepsilon(w) \leq 0$ if $s_k^p < 0$. Hence we can continue the chain of equality as follows:

$$\begin{aligned}
\ldots &= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) \\
&\quad + \int_{(V_k^\varepsilon)^{-1}((\inf I_k^p,z])} \operatorname{sign}(s_k^p)\ \tilde{r}_k\Big(\hat{u}^\varepsilon\big(V^\varepsilon(w)\big), \hat{v}_k^\varepsilon\big(V_k^\varepsilon(w)\big), \hat{\sigma}_k^\varepsilon\big(V_k^\varepsilon(w)\big)\Big)dw \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) \\
&\quad + \int_{(V_k^\varepsilon)^{-1}((\inf I_k^p,z])} \operatorname{sign}(s_k^p)\ \tilde{r}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(V_k^\varepsilon(w))\big), \hat{v}_k^\varepsilon\big(V_k^\varepsilon(w)\big), \hat{\sigma}_k^\varepsilon\big(V_k^\varepsilon(w)\big)\Big)dw \\
&\text{(making the change of variable } z = V_k^\varepsilon(w)) \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{\inf I_k^p}^z \operatorname{sign}(s_k^p)\ \tilde{r}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(\zeta)\big), \hat{v}_k^\varepsilon(\zeta), \hat{\sigma}_k^\varepsilon\zeta\Big)dz.
\end{aligned}$$

Now, if $s_k^p > 0$,

$$\begin{aligned}
\tilde{u}_k^p(z) &= \hat{u}^\varepsilon\big(\omega_k^\varepsilon(z)\big) \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{\inf I_k^p}^z \operatorname{sign}(s_k^p)\ \tilde{r}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(\zeta)\big), \hat{v}_k^\varepsilon(\zeta), \hat{\sigma}_k^\varepsilon(\zeta)\Big)d\zeta \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{\inf I_k^p}^z \tilde{r}_k\big(\tilde{u}_k^p(\zeta), \tilde{v}_k^p(\zeta), \tilde{\sigma}_k^p(\zeta)\big)d\zeta,
\end{aligned} \tag{5.54a}$$

while, if $s_k^p < 0$,

$$
\begin{aligned}
\tilde{u}_k^p(z) &= \hat{u}^\varepsilon\big(\omega_k^\varepsilon(-z)\big) \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{\inf I_k^p}^{-z} -\tilde{r}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(\zeta)\big), \hat{v}_k^\varepsilon(\zeta), \hat{\sigma}_k^\varepsilon(\zeta)\Big)d\zeta \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{-\inf I_k^p}^{z} \tilde{r}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(-\zeta)\big), \hat{v}_k^\varepsilon(-\zeta), \hat{\sigma}_k^\varepsilon(-\zeta)\Big)d\zeta \\
&= \int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) + \int_{-\inf I_k^p}^{z} \tilde{r}_k\big(\tilde{u}_k^p(\zeta), \tilde{v}_k^p(\zeta), \tilde{\sigma}_k^p(\zeta)\big)d\zeta.
\end{aligned}
\tag{5.54b}
$$

We already know, by Lemma 5.18, that, for some constant $a \in \mathbb{R}$,

$$
f_k^{\hat{\gamma}^\varepsilon}(z) = a + \int_{\inf I_k^p}^{z} \tilde{\lambda}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(\zeta)\big), \hat{v}_k^\varepsilon(\zeta), \hat{\sigma}_k^\varepsilon(\zeta)\Big)d\zeta.
$$

Hence. if $s_k^p > 0$,

$$
\begin{aligned}
f_k^{\hat{\gamma}^\varepsilon}(z) &= a + \int_{\inf I_k^p}^{z} \tilde{\lambda}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(\zeta)\big), \hat{v}_k^\varepsilon(\zeta), \hat{\sigma}_k^\varepsilon(\zeta)\Big)d\zeta \\
&= a + \int_{\inf I_k^p}^{z} \tilde{\lambda}_k\Big(\tilde{u}_k^p(z), v_k^p(z), \sigma_k^p(z)\Big) \\
\text{(by (5.53))} \quad &= \tilde{f}_k^p(z),
\end{aligned}
$$

while, if $s_k^p < 0$,

$$
\begin{aligned}
f_k^{\hat{\gamma}^\varepsilon}(-z) &= a + \int_{\inf I_k^p}^{-z} \tilde{\lambda}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(\zeta)\big), \hat{v}_k^\varepsilon(\zeta), \hat{\sigma}_k^\varepsilon(\zeta)\Big)d\zeta \\
&= a - \int_{-\inf I_k^p}^{-z} \tilde{\lambda}_k\Big(\hat{u}^\varepsilon\big(\omega_k^\varepsilon(-\zeta)\big), \hat{v}_k^\varepsilon(-\zeta), \hat{\sigma}_k^\varepsilon(-\zeta)\Big)d\zeta \\
&= a - \int_{-\inf I_k^p}^{z} \tilde{\lambda}_k\Big(\tilde{u}_k^p(z), v_k^p(z), \sigma_k^p(z)\Big) \\
\text{(by (5.53))} \quad &= -\tilde{f}_k^p(z).
\end{aligned}
$$

Recall that the reduced flux $\tilde{f}_k^p$ is defined up to an additive constant. Now, since

$$
(\hat{u}^\varepsilon, \hat{v}_1^\varepsilon, \dots, \hat{v}_N^\varepsilon, \hat{\sigma}_1^\varepsilon, \dots, \hat{\sigma}_N^\varepsilon)
$$

is the solution of the fixed point problem (5.17), if $s_k^p > 0$ and thus $\mathcal{S}_k^\varepsilon(x_p) \geq 0$,

$$
\begin{cases}
\tilde{v}_k^p(z) = \hat{v}_k^\varepsilon(z) = f_k^{\hat{\gamma}^\varepsilon}(z) - \operatorname*{conv}_{(\hat{\mathtt{x}}_k^\varepsilon(t))^{-1}(x_p)} f_k^{\hat{\gamma}^\varepsilon}(t)(z) = \tilde{f}_k^p(z) - \operatorname*{conv}_{I_k^p} \tilde{f}_k^p(z), \\
\tilde{\sigma}_k^p(z) = \hat{\sigma}_k(z) = \dfrac{d}{dz} \operatorname*{conv}_{(\hat{\mathtt{x}}_k^\varepsilon(t))^{-1}(x_p)} f_k^{\hat{\gamma}^\varepsilon}(z) = \dfrac{d}{dz} \operatorname*{conv}_{I_k^p} \tilde{f}_k^p(z).
\end{cases}
\tag{5.55a}
$$

Similarly, if $s_k^p < 0$ and thus $\mathcal{S}_k(x_p) < 0$,

$$
\begin{cases}
\tilde{v}_k^p(z) = \hat{v}_k^\varepsilon(-z) = -\Big[f_k^{\hat{\gamma}^\varepsilon}(-z) - \operatorname*{conv}_{(\hat{\mathtt{x}}_k^\varepsilon(t))^{-1}(x_p)} f_k^{\hat{\gamma}^\varepsilon}(t)(-z)\Big] = \tilde{f}_k^p(z) - \operatorname*{conc}_{I_k^p} \tilde{f}_k^p(z), \\
\tilde{\sigma}_k^p(z) = \hat{\sigma}_k(-z) = \dfrac{d}{dz} \operatorname*{conv}_{(\hat{\mathtt{x}}_k^\varepsilon(t))^{-1}(x_p)} f_k^{\hat{\gamma}^\varepsilon}(z) = \dfrac{d}{dz} \operatorname*{conc}_{I_k^p} \tilde{f}_k^p(z).
\end{cases}
\tag{5.55b}
$$

Hence, by (5.54) and (5.55), the curve $\tilde{\gamma}_k^p$ solves a Riemann problem of length s_k^p and starting point

$$\int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(dz).$$

To conclude the proof of the theorem, we need now to prove that

$$\int_{(0,\inf J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(dz) = u_{k-1}^p. \tag{5.56}$$

To avoid heavy notations, we assume that for all $p = 1,\dots,P$ and $k = 1,\dots,N$, $I_k^p \neq \emptyset$. The general case can be treated similarly. The proof of (5.56) is by induction of the set $\{(p,k) \mid p = 1,\dots,P,\ k = 1,\dots,N\}$ with the order

$$(p,k) \prec (p',k') \text{ if and only if } \big[p < p'\big] \text{ or } \big[p = p' \text{ and } k < k'\big].$$

For $p = k = 1$, (5.56) the l.h.s. of (5.56) is zero since the domain of integration is empty, while the r.h.s. is zero because $u^\varepsilon(t,\cdot)$ is identically zero out of a compact set. Assume now that (5.56) is proved for some (p,k). This implies that $(\tilde{u}_k^p, \tilde{v}_k^p, \tilde{\sigma}_k^p)$ is the exact curve solving the Riemann problem with length s_k^p and connecting u_{k-1}^p with u_k^p. Therefore, if $k < N$, by Lemma 5.15,

$$\begin{aligned}
\int_{(0,\inf J_{k+1}^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(dz) &= \int_{(0,\sup J_k^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(dz)\\
&= \hat{u}^\varepsilon\big(\sup J_k^p\big)\\
&= u_k^p.
\end{aligned}$$

Similarly, if $k = N$ and $p < P$,

$$\begin{aligned}
\int_{(0,\inf J_1^{p+1}]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(dz) &= \int_{(0,\sup J_N^p]} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(dz)\\
&= \hat{u}^\varepsilon\big(\sup J_N^p\big)\\
&= u_N^p\\
&= u_0^{p+1},
\end{aligned}$$

thus proving the inductive step and hence concluding the proof of the theorem. □

COROLLARY 5.30. *For every fixed time $t \in [0,T]$,*

(1) $D_x u^\varepsilon(t,\cdot) = \mathtt{x}^\varepsilon(t)_\sharp\big(\rho^\varepsilon(t) r^{\hat{\gamma}^\varepsilon(t)}\mathcal{L}^1|_{(L_0,L_N]}\big)$;
(2) for $\bar{\rho}^\varepsilon\mathcal{L}^1$-a.e. $w \in (L_0,L_N]$, $\bar{\sigma}^\varepsilon(t,w) = \hat{\sigma}^\varepsilon\big(t, V_k^\varepsilon(t,w)\big)$.

PROOF. Since $u^\varepsilon(t,\cdot)$ is piecewise constant and identically zero out of a compact set, its distributional derivative $D_x u^\varepsilon(t,\cdot)$ is a finite sum of Dirac's delta with size $u(t,x_p+) - u(t,x_p-)$, $p = 1,\dots,P$, where $\{x_p\}_p$ are the discontinuity points of $x \mapsto u^\varepsilon(t,x)$. Now, using the previous theorem, Lemma 5.15 and Proposition 5.11, we get (we do not explicitly write the dependence of the objects on time)

$$\begin{aligned}
u(t,x_p+) - u(t,x_p-) &= \hat{u}\big(\sup(\hat{\mathtt{x}}^\varepsilon)^{-1}(x_p)\big) - \hat{u}\big(\inf(\hat{\mathtt{x}}^\varepsilon)^{-1}(x_p)\big)\\
&= \int_{(\hat{\mathtt{x}}^\varepsilon)^{-1}(x_p)} V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta)\\
&= \hat{\mathtt{x}}_\sharp^\varepsilon V_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(x_p)\\
&= \mathtt{x}_\sharp^\varepsilon\Big(\rho^\varepsilon r^{\hat{\gamma}^\varepsilon}\mathcal{L}^1|_{(L_0,L_N]}\Big)(x_p),
\end{aligned}$$

thus proving the first part of the corollary.

The second part is an immediate consequence of the previous theorem, the definition of $\bar\sigma^\varepsilon$ in (3.31) and the definition of V_k^ε in (5.5). □

5.4.2. The interaction measure μ^ε. For every Glimm approximate solution u^ε, we introduce now the interaction measure μ^ε, defined as the sum of the amounts of interaction $\mathtt{A}(i\varepsilon, m\varepsilon)$ (see Definition 3.17) and we show that μ^ε can be used to bound

(1) the variation in time of the density function;
(2) the variation in time of the speed of the waves;
(3) the number of waves of different families which cross in a given area of the (t,x)-plane.

We heavily rely on the interaction estimate (3.52) proved in Chapter 3, without which it would be impossible to use μ^ε to bound such quantities.

Let us first show how the *transversal amount of interaction*, $\mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon)$ at any grid point $(i\varepsilon, m\varepsilon)$, $i \in \mathbb{N}$, $m \in \mathbb{Z}$ (see Definition 2.5), the *amount of creation* $\mathtt{A}^{\text{cr}}(i\varepsilon, m\varepsilon)$ and the *amount of cancellation* $\mathtt{A}^{\text{canc}}(i\varepsilon, m\varepsilon)$ (see Definition 2.8) can be rewritten using the density function $\rho^\varepsilon(t,w)$. We have

$$\mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) = \sum_{k>h} \iint_{\mathcal{W}^{\text{cross}}_{k,h}(i\varepsilon, m\varepsilon)} \bar\rho^\varepsilon\big((i-1/2)\varepsilon, w\big)\bar\rho^\varepsilon\big((i-1/2)\varepsilon, w'\big)dwdw'$$

where

$$\mathcal{W}^{\text{cross}}_{k,h}(i\varepsilon, m\varepsilon) := \bigg\{(w,w') \in (L_{k-1}, L_k] \times (L_{h-1}, L_h] \text{ such that} \\ \mathtt{x}^\varepsilon\big((i-1/2)\varepsilon, w\big) = (m-1/2)\varepsilon \text{ and } \mathtt{x}^\varepsilon\big((i-1/2)\varepsilon, w'\big) = m\varepsilon\bigg\}$$

and

$$\mathtt{A}^{\text{cr}}_k(i\varepsilon, m\varepsilon) = \int_{(\mathtt{x}^\varepsilon)^{-1}(i\varepsilon, m\varepsilon)\cap\mathcal{W}_k} \Big[\bar\rho^\varepsilon(i\varepsilon, w) - \bar\rho^\varepsilon((i-1)\varepsilon, w)\Big]^+,$$

$$\mathtt{A}^{\text{canc}}_k(i\varepsilon, m\varepsilon) = \int_{(\mathtt{x}^\varepsilon)^{-1}(i\varepsilon, m\varepsilon)\cap W_k} \Big[\bar\rho^\varepsilon(i\varepsilon, w) - \bar\rho^\varepsilon((i-1)\varepsilon, w)\Big]^-.$$

As a consequence

$$\mathtt{A}^{\text{cr}}_k(i\varepsilon, m\varepsilon) + \mathtt{A}^{\text{canc}}_k(i\varepsilon, m\varepsilon) = \int_{(\mathtt{x}^\varepsilon)^{-1}(i\varepsilon, m\varepsilon)\cap\mathcal{W}_k} \Big|\bar\rho^\varepsilon(i\varepsilon, w) - \bar\rho^\varepsilon((i-1)\varepsilon, w)\Big|. \tag{5.57}$$

Set also

$$\mathcal{W}^{\text{cross}}(i\varepsilon, m\varepsilon) := \bigcup_{k\neq h} \mathcal{W}^{\text{cross}}_{k,h}(i\varepsilon, m\varepsilon).$$

Notice that $\mathcal{W}^{\text{cross}}$ depends on the approximate solution u^ε under consideration.

Observe now that the change of speed of the waves located at grid point $(i\varepsilon, m\varepsilon)$ can be written as

$$\sum_{k=1}^N \Delta\sigma_k(i\varepsilon, m\varepsilon) = \int_{(\mathtt{x}^\varepsilon)^{-1}(i\varepsilon, m\varepsilon)} \big|\bar\rho^\varepsilon((i-1)\varepsilon, w)\bar\rho^\varepsilon(i\varepsilon, w)\big|\big|\bar\sigma^\varepsilon((i-1)\varepsilon, w) - \bar\sigma^\varepsilon(i\varepsilon, w)\big|dw$$

and thus, by Theorem 3.18,

$$\int_{(\mathtt{x}^\varepsilon)^{-1}(i\varepsilon, m\varepsilon)} \big|\bar\rho^\varepsilon((i-1)\varepsilon, w)\bar\rho^\varepsilon(i\varepsilon, w)\big|\big|\bar\sigma^\varepsilon((i-1)\varepsilon, w) - \bar\sigma^\varepsilon(i\varepsilon, w)\big|dw \leq \mathcal{O}(1)\mathtt{A}(i\varepsilon, m\varepsilon).$$

DEFINITION 5.31. The *interaction measure* related to the approximate solution u^ε is the finite positive Radon measure μ^ε on $[0,T]\times\mathbb{R}$ defined as

$$\mu^\varepsilon := \sum_{i,m} \mathtt{A}(i\varepsilon, m\varepsilon)\delta_{(i\varepsilon,m\varepsilon)},$$

where $\mathtt{A}(i\varepsilon, m\varepsilon)$ is the global amount of interaction at point $(i\varepsilon, m\varepsilon)$ introduced in Definition 3.17 and $\delta_{(t,x)}$ denotes the Dirac's delta at point (t,x).

Notice that, by Corollary 3.59, we know that the measure μ^ε is uniformly bounded, i.e.

$$\mu^\varepsilon\Big([0,T]\times\mathbb{R}\Big) \leq \mathcal{O}(1)\mathrm{Tot.Var.}(\bar{u}). \tag{5.58}$$

REMARK 5.32. The support of each measure μ^ε is included in a compact set which does not depend on ε, since the Glimm approximations u^ε are identically zero out of the set $[0,T]\times[-M-T, M+T]$, where $[-M,M]$ is the compact set such that $\bar{u}(x)=0$ for a.e. $x\notin[-M,M]$.

As we said at the beginning of this section, the interaction measure μ^ε can be used to bound the variation of the density function, the variation of the speed of the waves and the density of waves of different families which cross in a given area of the (t,x)-plane. This is done in the next three propositions. The first one concerns the variation of the density of the waves in a given time interval.

PROPOSITION 5.33. *Let $t_1, t_2, t \in [0,T]$, $t_1 \leq t \leq t_2$. Then it holds*

$$\int_{L_0}^{L_N} \mathrm{p.Tot.Var.}\Big(\bar{\rho}^\varepsilon(\cdot, w); [t_1,t_2]\Big)dw \leq \mu^\varepsilon\Big([t_1,t_2]\times\mathbb{R}\Big).$$

PROOF. The proof is an easy consequence of (5.57) and the definition of μ^ε. $\square$

The second proposition concerns the variation of the speed function in a given time interval.

PROPOSITION 5.34. *Let $t_1, t_2, t \in [0,T]$, $t_1 \leq t \leq t_2$. Then it holds*

$$\int_{L_0}^{L_N} \Big(\max_{\tau\in[t_1,t_2]} \bar{\sigma}^\varepsilon(\tau,w) - \min_{\tau\in[t_1,t_2]} \bar{\sigma}^\varepsilon(\tau,w)\Big)\bar{\rho}^\varepsilon(t,w)dw \leq \mathcal{O}(1)\mu^\varepsilon\Big([t_1,t_2]\times\mathbb{R}\Big).$$

PROOF. Set

$$\begin{aligned}
A &:= \big\{w\in(L_0,L_N] \;\big|\; |\rho^\varepsilon(t_1,w)| = |\rho^\varepsilon(t_2,w)| = 1\big\},\\
B_1 &:= \big\{w\in(L_0,L_N] \;\big|\; |\rho^\varepsilon(t,w)| = 1,\ \rho^\varepsilon(t_1,w) = 0\big\},\\
B_2 &:= \big\{w\in(L_0,L_N] \;\big|\; |\rho^\varepsilon(t,w)| = 1,\ \rho^\varepsilon(t_2,w) = 0\big\}.
\end{aligned}$$

Observe that

$$\big\{w\in(L_0,L_N] \;\big|\; |\rho^\varepsilon(t,w)| = 1\big\} = A\cup B_1\cup B_2.$$

We have

$$\begin{aligned}
\int_{B_1} \Big(\max_{t\in[t_1,t_2]} \bar{\sigma}^\varepsilon(t,w) - \min_{t\in[t_1,t_2]} \bar{\sigma}^\varepsilon(t,w)\Big)\bar{\rho}^\varepsilon(t,w)dw &\leq \int_{B_1} \bar{\rho}^\varepsilon(t,w)dw\\
&= \int_{B_1} \bar{\rho}^\varepsilon(t,w) - \bar{\rho}^\varepsilon(t_1,w)dw\\
&\leq \int_{B_1} \mathrm{p.Tot.Var.}\Big(\bar{\rho}^\varepsilon(\cdot,w);[t_1,t_2]\Big)dw\\
\text{(by the first part of the proposition)} &\leq \mu^\varepsilon\Big([t_1,t_2]\times\mathbb{R}\Big).
\end{aligned} \tag{5.59}$$

Similarly,

$$\int_{B_1} \Big(\max_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) - \min_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) \Big) \bar\rho^\varepsilon(t,w) dw \leq \mu^\varepsilon\Big([t_1,t_2]\times\mathbb{R}\Big). \tag{5.60}$$

Set now

$$\mathcal{I} := \big\{ i \in \mathbb{N} \;\big|\; i\varepsilon \in [t_1,t_2] \big\}.$$

It holds

$$\begin{aligned}
&\int_A \Big(\max_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) - \min_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) \Big) \bar\rho^\varepsilon(t,w) dw \\
&\quad = \int_A \Big(\max_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) - \min_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) \Big) dw \\
&\quad \leq \int_A \text{p.Tot.Var.}\Big(\bar\sigma^\varepsilon(\cdot,w);[t_1,t_2]\Big) dw \\
&\quad = \sum_{i\in\mathcal{I}} \int_A \big|\bar\sigma^\varepsilon((i-1)\varepsilon,w) - \bar\sigma^\varepsilon(i\varepsilon,w)\big| dw \\
&\quad \leq \sum_{i\in\mathcal{I}} \int_{L_0}^{L_N} \big|\bar\rho^\varepsilon((i-1)\varepsilon,w)\bar\rho^\varepsilon(i\varepsilon,w)\big| \big|\bar\sigma^\varepsilon((i-1)\varepsilon,w) - \bar\sigma^\varepsilon(i\varepsilon,w)\big| dw \\
&\quad = \sum_{i\in\mathcal{I}} \sum_{m\in\mathbb{Z}} \int_{(\mathtt{x}^\varepsilon(i\varepsilon))^{-1}(m\varepsilon)} \big|\bar\rho^\varepsilon((i-1)\varepsilon,w)\bar\rho^\varepsilon(i\varepsilon,w)\big| \big|\bar\sigma^\varepsilon((i-1)\varepsilon,w) - \bar\sigma^\varepsilon(i\varepsilon,w)\big| dw \\
&\quad \leq \mathcal{O}(1) \sum_{i\in\mathcal{I}} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \\
&\quad = \mathcal{O}(1)\mu^\varepsilon\Big([t_1,t_2]\times\mathbb{R}\Big).
\end{aligned} \tag{5.61}$$

Therefore

$$\begin{aligned}
&\int_{L_0}^{L_N} \Big(\max_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) - \min_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) \Big) \bar\rho^\varepsilon(t,w) dw \\
&\qquad \leq \Big(\int_A + \int_{B_1} + \int_{B_2} \Big) \Big(\max_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) - \min_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) \Big) \bar\rho^\varepsilon(t,w) dw \\
&\text{(by (5.59), (5.60), (5.61))} \leq \mathcal{O}(1)\mu^\varepsilon\Big([t_1,t_2]\times\mathbb{R}\Big),
\end{aligned}$$

which is what we wanted to prove. □

Before stating the third and last proposition of this section which concerns the number of waves of different families which cross in a given area of the (t,x)-plane, we state and prove the following two lemmas. The first lemma estimate the distance between the position map $\mathtt{x}^\varepsilon$ and the integral over a time interval of the speed map.

LEMMA 5.35. *Let T be a fixed time. For every $\eta > 0$, there is $\bar\varepsilon > 0$ such that for every $0 < \varepsilon \leq \bar\varepsilon$, for every $w \in (L_0, L_N]$ and for every $t_1, t_2 \in [0,T]$, if $\bar\rho^\varepsilon(t_1,w) = \bar\rho^\varepsilon(t_2,w) = 1$ and if $t_2 - t_1 > \eta$, then*

$$\begin{aligned}
&\left| \mathtt{x}^\varepsilon(t_2,w) - \Big(\mathtt{x}^\varepsilon(t_1,w) + \int_{t_1}^{t_2} \bar\sigma^\varepsilon(t,w)dt \Big) \right| \\
&\qquad \leq 2C(t_2-t_1)\Big[\eta + \max_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w) - \min_{t\in[t_1,t_2]} \bar\sigma^\varepsilon(t,w)\Big].
\end{aligned} \tag{5.62}$$

PROOF. Fix $T > 0$ and $\eta > 0$. There exists $\bar{n}$ such that for every $n \geq \bar{n}$,

$$\frac{1+\log n}{n} \leq \frac{\eta}{2}.$$

Define

$$\bar{\varepsilon} := \min\left\{\frac{\eta}{\bar{n}+2}, \frac{C\eta^2}{4}\right\}. \tag{5.63}$$

We have now to prove that for every $0 < \varepsilon \leq \bar{\varepsilon}$ and for every $w \in \mathcal{W}$ and for every $t_1, t_2 \in [0, T]$, if $t_2 - t_1 > \eta$, estimate (5.62) holds. Hence fix $0 < \varepsilon < \bar{\varepsilon}$, fix $w \in \mathcal{W}$, fix $t_1, t_2 \in [0, T]$ such that $\bar{\rho}^\varepsilon(t_1, w) = \bar{\rho}^\varepsilon(t_2, w) = 1$ and $t_2 - t_1 > \eta$. Define

$$i_1 := \min\{i \in \mathbb{N} \mid i\varepsilon \geq t_1\}, \qquad i_2 := \max\{i \in \mathbb{N} \mid i\varepsilon \leq t_2\}.$$

Notice that $(i_2 - i_1)\varepsilon \geq \eta - 2\varepsilon$ and thus, by our choice of $\bar{\varepsilon}$, $i_2 - i_1 \geq \bar{n}$; hence

$$\frac{1+\log(i_2 - i_1)}{i_2 - i_1} \leq \frac{\eta}{2}. \tag{5.64}$$

We can now use Lemma 4.5 to conclude the proof as follows:

$$\begin{aligned}
&\left|\mathtt{x}^\varepsilon(t_2, w) - \left(\mathtt{x}^\varepsilon(t_1, w) + \int_{t_1}^{t_2} \bar{\sigma}^\varepsilon(t, w)dt\right)\right| \\
&\qquad \leq \left|\mathtt{x}^\varepsilon(t_2, w) - \mathtt{x}^\varepsilon(i_2\varepsilon, w)\right| + \left|\mathtt{x}^\varepsilon(i_2\varepsilon, w) - \left(\mathtt{x}^\varepsilon(i_1\varepsilon, w) + \varepsilon\sum_{i=i_1}^{i_2} \bar{\sigma}^\varepsilon(i\varepsilon, w)\right)\right| \\
&\qquad\qquad + \left|\mathtt{x}^\varepsilon(i_1\varepsilon, w) - \mathtt{x}^\varepsilon(t_1, w)\right| + \left|\int_{i_1\varepsilon}^{i_2\varepsilon} \bar{\sigma}^\varepsilon(t, w)dt - \int_{t_1}^{t_2} \bar{\sigma}^\varepsilon(t, w)dt\right| \\
&\qquad \leq 4\varepsilon + 2C(t_2 - t_1)\left[\max_{t\in[t_1,t_2]} \bar{\sigma}^\varepsilon(t, w) - \min_{t\in[t_1,t_2]} \bar{\sigma}^\varepsilon(t, w) + \frac{\eta}{2}\right] \\
\text{(by (5.63))} &\qquad \leq 2C(t_2 - t_1)\left[\max_{t\in[t_1,t_2]} \bar{\sigma}^\varepsilon(t, w) - \min_{t\in[t_1,t_2]} \bar{\sigma}^\varepsilon(t, w) + \eta\right].
\end{aligned}$$

□

This second lemma shows that the time at which two waves of different families cross is proportional to their distance at a given time $\bar{t}$.

LEMMA 5.36. *Let $k < k'$ be two families. For every*

$$0 < \eta \leq \min_{h=1,\dots,N-1} \left|\lambda_{h+1}^{\min} - \lambda_h^{\max}\right|$$

there is $\bar{\varepsilon} > 0$ such that for every $0 < \varepsilon \leq \bar{\varepsilon}$, for every $w \in \mathcal{W}_k$, $w' \in \mathcal{W}_{k'}$, with $\rho^\varepsilon(\bar{t}, w) = \rho^\varepsilon(t, w) \neq 0$, $\rho^\varepsilon(\bar{t}, w') = \rho^\varepsilon(t, w') \neq 0$, for every $\bar{t}, t \in [0, T]$:

a) if $\mathtt{x}^\varepsilon(\bar{t}, w') \leq \mathtt{x}^\varepsilon(\bar{t}, w)$ and

$$t - \bar{t} > \frac{\mathtt{x}^\varepsilon(\bar{t}, w) - \mathtt{x}^\varepsilon(\bar{t}, w') + \eta}{\lambda_{k'}^{\min} - \lambda_k^{\max} - \eta}, \tag{5.65}$$

then $\mathtt{x}^\varepsilon(t, w) < \mathtt{x}^\varepsilon(t, w')$;

b) if $\mathtt{x}^\varepsilon(\bar{t}, w) \leq \mathtt{x}^\varepsilon(\bar{t}, w')$ and

$$\bar{t} - t > \frac{\mathtt{x}^\varepsilon(\bar{t}, w') - \mathtt{x}^\varepsilon(\bar{t}, w) + \eta}{\lambda_{k'}^{\min} - \lambda_k^{\max} - \eta},$$

then $\mathtt{x}^\varepsilon(t, w') < \mathtt{x}^\varepsilon(t, w)$.

PROOF. We prove just Point a), the proof of Point b) being similar. Set for simplicity

$$\Delta \mathtt{x}^\varepsilon := \mathtt{x}^\varepsilon(\bar{t}, w) - \mathtt{x}^\varepsilon(\bar{t}, w'), \qquad \Delta t := t - \bar{t}, \qquad \Delta\lambda := \lambda_{k'}^{\min} - \lambda_k^{\max}.$$

Fix $\eta > 0$. There is $\bar{n} \in \mathbb{N}$ such that for every $n \geq \bar{n}$,

$$C\frac{1 + \log n}{n} \leq \eta,$$

where C is the constant which appear in (2.15). Define

$$\bar{\varepsilon} := \min\left\{\frac{\eta}{6}, \frac{1}{\bar{n}+2} \cdot \frac{\Delta t + \eta}{\Delta \mathtt{x} - \eta}\right\}.$$

Take any $0 < \varepsilon \leq \bar{\varepsilon}$ and consider the Glimm approximate solution u^ε. Define also

$$i_1 := \min\{i \in \mathbb{N} \mid i\varepsilon \geq \bar{t}\}, \qquad i_2 := \max\{i \in \mathbb{N} \mid i\varepsilon \leq t\}.$$

Notice that

$$(i_2 - i_1)\varepsilon \geq t - \bar{t} - 2\varepsilon \geq \frac{\Delta t + \eta}{\Delta \mathtt{x} - \eta} - \varepsilon$$

and thus, by our choice of $\bar{\varepsilon}$, it holds $i_2 - i_1 \geq \bar{n}$. Since w belongs to the k-th family, w' belongs to the k'-th family and both have non-zero density in the interval $[\bar{t}, t]$, whenever the value $\vartheta_i \in (\lambda_k^{\max}, \lambda_{k'}^{\min})$, by Point (5) in the definition of wave tracing in Section 3.3.1, we have that

$$\mathtt{x}^\varepsilon((i+1)\varepsilon, w) = \mathtt{x}^\varepsilon(i\varepsilon, w), \qquad \mathtt{x}^\varepsilon((i+1)\varepsilon, w') = \mathtt{x}^\varepsilon(i\varepsilon, w') + \varepsilon$$

and thus

$$\mathtt{x}^\varepsilon(i_2\varepsilon, w) - \mathtt{x}^\varepsilon(i_2\varepsilon, w') \leq \mathtt{x}^\varepsilon(i_1\varepsilon, w) - \mathtt{x}^\varepsilon(i_1\varepsilon, w') - \varepsilon\sharp\big\{i \in [i_1, i_2 - 1] \bigm| \vartheta_i \in (\lambda_k^{\max}, \lambda_{k'}^{\min})\big\}.$$

Therefore

$$\begin{aligned}
\mathtt{x}^\varepsilon(t, w) - \mathtt{x}^\varepsilon(t, w') &= 2\varepsilon + \mathtt{x}^\varepsilon(i_2\varepsilon, w) - \mathtt{x}^\varepsilon(i_2\varepsilon, w') \\
&\leq 2\varepsilon + \mathtt{x}^\varepsilon(i_1\varepsilon, w) - \mathtt{x}^\varepsilon(i_1\varepsilon, w') - \varepsilon\sharp\big\{i \in [i_1, i_2 - 1] \bigm| \vartheta_i \in (\lambda_k^{\max}, \lambda_{k'}^{\min})\big\} \\
&\leq 4\varepsilon + \Delta\mathtt{x}^\varepsilon - \varepsilon(i_2 - i_1)\frac{\sharp\big\{i \in [i_1, i_2 - 1] \bigm| \vartheta_i \in (\lambda_k^{\max}, \lambda_{k'}^{\min})\big\}}{i_2 - i_1} \\
&\leq 6\varepsilon + \Delta\mathtt{x}^\varepsilon - \Delta t \cdot \frac{\sharp\big\{i \in [i_1, i_2 - 1] \bigm| \vartheta_i \in (\lambda_k^{\max}, \lambda_{k'}^{\min})\big\}}{i_2 - i_1} \\
&\quad \text{(since the sequence } \{\vartheta_i\}_i \text{ satisfies (2.15))} \\
&\leq (\eta + \Delta\mathtt{x}^\varepsilon) - \Delta t(\Delta\lambda - \eta) \\
\text{(by (5.65))} \quad &\leq 0,
\end{aligned}$$

which is what we wanted to get. The proof of Point b) is similar. $\square$

We can now state the third and last proposition of this section, which estimate the number of pairs of waves of different families which at a given time $\bar{t}$ have distance less than $\delta > 0$ in terms of the interaction measure μ^ε.

PROPOSITION 5.37. *For every*

$$0 < \eta \leq \min_{h=1,\dots,N-1} \left|\lambda_{h+1}^{\min} - \lambda_h^{\max}\right|$$

there exists $\bar{\varepsilon} > 0$ such that for every $0 < \varepsilon \leq \bar{\varepsilon}$, the following holds. Fix $\delta > 0$ and $\bar{t} \in [0, T]$. Define the set

$$E_\delta^\varepsilon := \big\{(w, w') \in \mathcal{W}_k \times \mathcal{W}_h \bigm| k \neq h \text{ and } |\mathtt{x}^\varepsilon(\bar{t}, w) - \mathtt{x}^\varepsilon(\bar{t}, w')| < \delta\big\}.$$

Then

$$\iint_{E^\varepsilon_\delta} \bar\rho^\varepsilon(\bar t, w)\bar\rho^\varepsilon(\bar t, w')dwdw' \le \mathcal{O}(1)\mu^\varepsilon\left(\left[\bar t - \frac{\eta+\delta}{\Lambda-\eta}, \bar t + \frac{\eta+\delta}{\Lambda-\eta}\right] \times \mathbb{R}\right),$$

where, as before, $\Lambda := \min_{k,h} \left|\lambda^{\max}_k - \lambda^{\min}_h\right|$.

Proof. Fix $\eta > 0$. Let $\bar\varepsilon$ be give by Lemma 5.36. Take any $0 < \varepsilon \le \bar\varepsilon$ and $\delta > 0$. Set, for simplicity,

$$I := \left[\bar t - \frac{\eta+\delta}{\Lambda-\eta}, \bar t + \frac{\eta+\delta}{\Lambda-\eta}\right].$$

Define

$$A^\varepsilon_\delta := \left\{(w,w') \in E^\varepsilon_\delta \;\middle|\; \rho^\varepsilon(t) = \rho^\varepsilon(\bar t) \text{ for every } t \in I\right\}, \qquad B^\varepsilon_\delta := E^\varepsilon_\delta \setminus A^\varepsilon_\delta.$$

By Lemma 5.36, we know that if $(w,w') \in A^\varepsilon_\delta$ and $\bar\rho^\varepsilon(t,w) = 1$ for every time $t \in I$, then w, w' must cross in the time interval I and thus

$$A^\varepsilon_\delta \subseteq \bigcup_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \mathcal{W}^{\text{cross}}(i\varepsilon, m\varepsilon),$$

where

$$\mathcal{I} := \left\{i \in \mathbb{N} \;\middle|\; \bar t - \frac{\delta+\eta}{\Lambda-\eta} < i\varepsilon < \bar t + \frac{\delta+\eta}{\Lambda-\eta}\right\}.$$

Therefore

$$\begin{aligned}
&\iint_{A^\varepsilon_\delta} \bar\rho^\varepsilon(\bar t, w)\bar\rho^\varepsilon(\bar t, w')dwdw' \\
&\quad= \sum_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \iint_{A^\varepsilon_\delta \cap \mathcal{W}^{\text{cross}}(i\varepsilon,m\varepsilon)} \bar\rho^\varepsilon(\bar t, w)\bar\rho^\varepsilon(\bar t, w')dwdw' \\
&\quad= \sum_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \iint_{A^\varepsilon_\delta \cap \mathcal{W}^{\text{cross}}(i\varepsilon,m\varepsilon)} \Big[\bar\rho^\varepsilon(\bar t, w) - \bar\rho^\varepsilon\big((i-1)\varepsilon, w\big)\Big]\bar\rho^\varepsilon(\bar t, w')dwdw' \\
&\qquad\qquad + \sum_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \iint_{A^\varepsilon_\delta \cap \mathcal{W}^{\text{cross}}(i\varepsilon,m\varepsilon)} \bar\rho^\varepsilon\big((i-1)\varepsilon, w\big)\Big[\bar\rho^\varepsilon(\bar t, w') - \bar\rho^\varepsilon\big((i-1)\varepsilon, w'\big)\Big]dwdw' \\
&\qquad\qquad + \sum_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \iint_{A^\varepsilon_\delta \cap \mathcal{W}^{\text{cross}}(i\varepsilon,m\varepsilon)} \bar\rho^\varepsilon\big((i-1)\varepsilon, w\big)\bar\rho^\varepsilon\big((i-1)\varepsilon, w'\big)dwdw' \\
&\quad\le \sum_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \iint_{A^\varepsilon_\delta \cap \mathcal{W}^{\text{cross}}(i\varepsilon,m\varepsilon)} \text{p.Tot.Var.}\big(\bar\rho^\varepsilon(\cdot, w); I\big)dwdw' \\
&\qquad\qquad + \sum_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \iint_{A^\varepsilon_\delta \cap \mathcal{W}^{\text{cross}}(i\varepsilon,m\varepsilon)} \text{p.Tot.Var.}\big(\bar\rho^\varepsilon(\cdot, w'); I\big)dwdw' + \sum_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) \\
&\quad\le \iint_{A^\varepsilon_\delta} \text{p.Tot.Var.}\big(\bar\rho^\varepsilon(\cdot, w); I\big)dwdw' + \sum_{\substack{i\in\mathcal{I}\\ m\in\mathbb{Z}}} \mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) \\
&\quad\le 2\mu^\varepsilon\big(I \times \mathbb{R}\big),
\end{aligned}$$

where in the last inequality we have used the Proposition 5.33 and the definition of μ^ε. The conclusion follows easily observing that, by Proposition 5.33, $\mathcal{L}^2(B^\varepsilon_\delta) \leq 2\mu^\varepsilon(I \times \mathbb{R})$. $\square$

5.5. Convergence of the position and the density

In this section we start the proof of Theorem C, proving Step 2 and Step 3 in the sketch of its proof provided in Section 5.2. In particular we will show that there exist three maps

$$\begin{aligned}
&\mathtt{x} : [0,T] \times (L_0, L_N] \to \mathbb{R} \text{ the } \textit{position function},\\
&\rho : [0,T] \times (L_0, L_N] \to [-1,1] \text{ the } \textit{density function},\\
&\bar\rho : [0,T] \times (L_0, L_N] \to [-1,1] \text{ the } \textit{absolute density function},
\end{aligned}$$

which, together with the numbers $L_0 \leq \cdots L_N$ already introduced in Step 1 in the sketch of the proof of Theorem C in Section 5.2, will be the candidate Lagrangian representation. Moroever, we will prove that Property (a) and (d) in the definition of Lagrangian representation (Definition 5.21) hold.

We know by Theorem B that for all time $t \in [0,T]$ the Glimm approximate solution $u^\varepsilon(t,\cdot) \to S_t\bar u$ in L^1 as $\varepsilon \to 0$. Since we prefer to work with sequences, rather than with the whole family of approximations $\{u^\varepsilon\}_\varepsilon$, take a sequence $\varepsilon_n \to 0$ as $n \to \infty$ and set for simplicity $u^n := u^{\varepsilon_n}$. We relabel now

$$L^n_k := L^{\varepsilon_n}_k, \text{ for } k = 1, \dots, N,$$

and

$$\mathtt{x}^n := \mathtt{x}^{\varepsilon_n}, \qquad \rho^n := \rho^{\varepsilon_n}, \qquad \bar\rho^n := \bar\rho^{\varepsilon_n}, \qquad \bar\sigma^n := \bar\sigma^{\varepsilon_n}$$

and the same for all related objects

$$\mathcal{S}^n_k,\ \mathfrak{R}^n,\ M^n_k,\ M^n,\ V^n_k,\ V^n,\ \hat{\mathtt{x}}^n_k,\ \hat{\mathtt{x}}^n,\ \hat\gamma^n,\ \hat u^n,\ \hat v^n_k,\ \hat\sigma^n_k,$$

constructed with the techniques introduced in Section 5.1 and for the interaction measure $\mu^n := \mu^{\varepsilon_n}$ introduced in Section 5.4. We prove first the convergence of the position functions.

Proposition 5.38. *There exists a subsequence of $(\mathtt{x}^n)_n$, still denoted by $(\mathtt{x}^n)_n$ and a map*

$$\mathtt{x} : [0,T] \times (L_0, L_N]$$

such that

(1) *for every time $t \in [0,T]$*

$$\big\|\mathtt{x}^n(t,\cdot) - \mathtt{x}(t,\cdot)\big\|_{L^1((L_0,L_N])} \to 0;$$

(2) *for every time $t \in [0,T]$ the map $w \mapsto \mathtt{x}(t,w)$ is increasing on each $(L_{k-1}, L_k]$;*
(3) *for $\mathcal{L}^1$-a.e. wave $w \in (L_0, L_N]$, the map $t \mapsto \mathtt{x}(t,w)$ is 1-Lipschitz.*

As an immediate consequence of the previous proposition, we get that Property (a) in the definition of Lagrangian representation, Definition 5.21, holds.

Proof. Let D be a dense subset of $[0,T]$. For every $t \in D$, the family of maps

$$\big\{w \mapsto \mathtt{x}^n(t,w)\big\}_{n\in\mathbb{N}}$$

is compact in L^1 by Proposition 1.42 and the fact that any $\mathtt{x}^n(t)$ is increasing when restricted to each $(L_{k-1}, L_k]$. Hence, by a diagonal argument, we can extract a subsequence (still denoted by $\mathtt{x}^n$) and for every $t \in D$ a map $w \mapsto \mathtt{x}(t,w)$, increasing when restricted to each $(L_{k-1}, L_k]$, such that

$$\big\|\mathtt{x}^n(t,\cdot) - \mathtt{x}(t,\cdot)\big\|_{L^1((L_0,L_N])} \to 0 \ \text{ for every } t \in D. \tag{5.66}$$

We thus have a set $E \subseteq (L_0, L_N]$ such that $\mathcal{L}^1(E) = 0$ and

$$|\mathtt{x}^n(t,w) - \mathtt{x}(t,w)| \to 0 \text{ as } n \to \infty \text{ for every } w \notin E \text{ and for every } t \in D.$$

The 1-Lipschitz continuity of the maps $t \mapsto \mathtt{x}^n(t,w)$ implies that for every $w \notin E$ the map

$$D \ni t \mapsto \mathtt{x}(t,w)$$

is Lipschitz and thus it can be extended to a 1-Lipschitz map $[0,T] \ni t \mapsto \mathtt{x}(t,w)$. We have thus defined a map $\mathtt{x} : [0,T] \times \big((L_0, L_N] \setminus E\big) \to \mathbb{R}$. We prove now that Properties (1), (2), (3) in the statement hold.

(1) Fix any time $t \in [0,T]$. Since D is dense in $[0,T]$, for every $\eta > 0$ there exists $\tilde{t} \in D$ such that $|t - \tilde{t}| \leq \eta$. We thus have

$$\begin{aligned}
\big\|\mathtt{x}^n(t,\cdot) - \mathtt{x}(t,\cdot)\big\|_1 &\leq \big\|\mathtt{x}^n(t,\cdot) - \mathtt{x}(\tilde{t},\cdot)\big\|_1 + \big\|\mathtt{x}^n(\tilde{t},\cdot) - \mathtt{x}(\tilde{t},\cdot)\big\|_1 + \big\|\mathtt{x}(\tilde{t},\cdot) - \mathtt{x}(t,\cdot)\big\|_1 \\
&\leq 2|\tilde{t} - t| + \big\|\mathtt{x}^n(\tilde{t},\cdot) - \mathtt{x}(\tilde{t},\cdot)\big\|_1 \\
&\leq \eta + \big\|\mathtt{x}^n(\tilde{t},\cdot) - \mathtt{x}(\tilde{t},\cdot)\big\|_1.
\end{aligned}$$

Taking the limit as $n \to \infty$, using (5.66) and by the arbitrary of η we get Point (1).

(2) As before, fix $t \in [0,T]$, fix $\eta > 0$ and take $\tilde{t} \in D$ such that $|\tilde{t} - t| \leq \eta$. Take any subset $S \subseteq (L_{k-1}, L_k]^2$. We have

$$\begin{aligned}
&\iint_S \big(\mathtt{x}(t,w) - \mathtt{x}(t,w')\big)(w - w')dwdw' \\
&\quad = \iint_S \big(\mathtt{x}(t,w) - \mathtt{x}(\tilde{t},w)\big)(w - w')dwdw' + \iint_S \big(\mathtt{x}(\tilde{t},w) - \mathtt{x}(\tilde{t},w')\big)(w - w')dwdw' \\
&\qquad\quad + \iint_S \big(\mathtt{x}(\tilde{t},w') - \mathtt{x}(t,w')\big)(w - w')dwdw' \\
&\quad \leq 2(L_k - L_{k-1})^3|\tilde{t} - t| + \iint_S \big(\mathtt{x}(\tilde{t},w) - \mathtt{x}(\tilde{t},w')\big)(w - w')dwdw'
\end{aligned}$$

(since the map $w \mapsto \mathtt{x}(\tilde{t},w)$ is increasing on $(L_{k-1}, L_k]$)

$$2(L_k - L_{k-1})^3\eta.$$

By the arbitrary of $\eta > 0$ we get Point (2).

(3) Finally, for every $w \notin E$, the fact that $t \mapsto \mathtt{x}(t,w)$ is 1-Lipschitz is an easy consequence of its definition. □

We prove now the convergence of the density functions.

Proposition 5.39. *There exists a subsequence of $(\rho^n)_n$, still denoted by $(\rho^n)_n$ and two maps*

$$\rho : [0,T] \times (L_0, L_N] \to [-1,1], \qquad \bar{\rho} : [0,T] \times (L_0, L_N] \to [0,1],$$

such that, up to a countable set $\mathcal{Z}_1$ of times:

(1) $\rho^n(t,\cdot)$ converges to $\rho(t,\cdot)$ weakly in $L^\infty((L_0, L_N])$;*
(2) $\bar{\rho}^n(t,\cdot)$ converges to $\bar{\rho}(t,\cdot)$ weakly in $L^\infty((L_0, L_N])$;*
(3) $|\rho(t,w)| \leq \bar{\rho}(t,w)$ for a.e. wave $w \in (L_0, L_N]$;
(4) extending on the whole $\mathbb{R}^2$ the maps $\rho(t,w), \bar{\rho}(t,w)$ to zero outside the set $[0,T] \times (L_0, L_N]$, the distributions $D_t\rho$ and $D_t\bar{\rho}$ are finite Radon measure on $\mathbb{R}^2$.

As an immediate consequence of the previous proposition, we get that Property (d) in the definition of Lagrangian representation, Definition 5.21, holds.

PROOF. We start with the proof of Point (1). Fix a small parameter $\eta > 0$ and assume that each ρ^n is defined on the open set $\Omega := (-\eta, T+\eta)\times(L_0-\eta, L_N+\eta)$, setting $\rho^n(t,w) = 0$ if $(t,w) \notin [0,T] \times (L_0, L_N]$. Define now, for every fixed $n \in \mathbb{N}$ and for every fixed $w \in (L_0 - \eta, L_N + \eta)$, the finite Radon measure on $(-\eta, T+\eta)$

$$\nu_w^n := D_t\rho^n(\cdot, w),$$

where $D_t\rho^n(\cdot,w)$ denotes the distributional derivative of the map $(-\eta, T+\eta) \ni t \mapsto \rho^n(t,w)$ for every fixed w. Since, for every fixed wave w, the map $\rho^n(t,w)$ as a function of time takes values in the set $\{-1, 0\}$ (if w is negative) or in the set $\{0,1\}$ (if w is positive) (see Point (4) in the definition of wave tracing at page 52), it is easy to see that

$$|\nu_w^n|\Big((-\eta, T+\eta)\Big) \leq 2.$$

Set

$$\nu^n := \bigg(\int_{L_0-\eta}^{L_N+\eta} \nu_w^n dw\bigg)$$

(see Definition 1.31) and observe that

$$|\nu^n|(\Omega) \leq 2(L_N - L_0). \tag{5.67}$$

Therefore, by Theorem 1.27, we can find two measures ν, $\bar{\nu}$ on Ω and a subsequence (still denoted by $(\nu^n)_n$) such that (ν^n) converges weakly* to ν and $(|\nu^n|)$ converges weakly* to $\bar{\nu}$. We want to disintegrate the measure ν on Ω w.r.t. the Lebesgue measure on $(L_0-\eta, L_N+\eta)$. We thus have to prove first that

$$(\mathrm{Pr}_w)_\sharp|\nu| \ll \mathcal{L}^1|_{(L_0-\eta, L_N+\eta)}$$

(where $\mathrm{Pr}_w(t,w) = w$ is the standard projection). This follows quite easily from the fact that $\mathrm{supp}\big(|\nu^n|\big) \subseteq [\eta/2, T+\eta/2] \times [L_0 - \eta/2, L_N + \eta/2]$ and the from the fact, for the approximations, the density functions $d^n(w)$ defined by

$$(\mathrm{Pr}_w)_\sharp|\nu^n| = d^n(w)\mathcal{L}^1|_{(L_0-\eta, L_N+\eta)},$$

are uniformly bounded by 2 and then using Lemmas 1.20, 1.28 and 1.29. Hence we can apply the Disintegration Theorem (see Theorem 1.32) to find a family $\{\nu_w\}_w$ of Radon measure on $(-\eta, T_\eta)$ such that

$$\nu = \int_{L_0-\eta}^{L_N+\eta} \nu_w dw.$$

For a.e. $w \in (L_0, L_N]$, we can thus define

$$\rho(t,w) := \nu_w\Big((-\eta, t)\Big).$$

We want now to prove that, up to a countable set $\tilde{Z}$ of times, for every fixed time $t \in [0,T]$, the maps $\rho^n(t,\cdot)$ converge weak* to the map $\rho(t,\cdot)$. First, define

$$\mathcal{Z}_1 := \Big\{t \in (\eta, T+\eta) \;\Big|\; (\mathrm{Pr}_t)_\sharp\bar{\nu}\big(\{t\}\big) > 0\Big\}. \tag{5.68}$$

Clearly $\mathcal{Z}_1$ is countable. Fix now any $\bar{t} \notin \mathcal{Z}_1$. Let $\phi \in L^1((L_0, L_N])$. Using the properties of the generalized product of measures (see Definition 1.31), we have

$$\begin{aligned}\int_{L_0}^{L_N} \phi(w)\rho^n(\bar{t}, w)dw &= \int_{L_0-\eta}^{L_N+\eta} \phi(w)\nu_w^n\Big((-\eta, \bar{t})\Big)dw \\ &= \int_{L_0-\eta}^{L_N+\eta} \phi(w)\Big(\int_{-\eta}^{T+\eta} \chi_{(\eta,\bar{t})}(t)\nu_w^n(dt)\Big)dw \\ &= \int_{L_0-\eta}^{L_N+\eta}\int_{-\eta}^{T+\eta} \phi(w)\chi_{(\eta,\bar{t})}(t)\nu^n(dtdw).\end{aligned}$$

We want to apply Proposition 1.30 to pass to the limit the r.h.s. of the last equation. Notice that the set of discontinuity point of the map $(t, w) \mapsto \phi(w)\chi_{(\eta,\bar{t})}(t)$ is contained in $\{\bar{t}\} \times (L_0 - \eta, L_N + \eta)$ and, since $\bar{t} \notin \mathcal{Z}_1$, $\bar{\nu}\{\bar{t}\} \times (L_0 - \eta, L_N + \eta) = 0$. Therefore by Proposition 1.30

$$\int_{L_0-\eta}^{L_N+\eta}\int_{-\eta}^{T+\eta} \phi(w)\chi_{(\eta,\bar{t})}(t)\nu^n(dtdw) \to \int_{L_0-\eta}^{L_N+\eta}\int_{-\eta}^{T+\eta} \phi(w)\chi_{(\eta,\bar{t})}(t)\nu(dtdw).$$

Notice now that

$$\begin{aligned}\int_{L_0-\eta}^{L_N+\eta}\int_{-\eta}^{T+\eta} \phi(w)\chi_{(\eta,\bar{t})}(t)\nu(dtdw) &= \int_{L_0-\eta}^{L_N+\eta} \phi(w)\Big(\int_{-\eta}^{T+\eta} \chi_{(\eta,\bar{t})}(t)\nu_w(dt)\Big)dw \\ &= \int_{L_0-\eta}^{L_N+\eta} \phi(w)\nu_w\Big((-\eta, \bar{t})\Big)dw \\ &= \int_{L_0}^{L_N} \phi(w)\rho(\bar{t}, w)dw,\end{aligned}$$

which proves that $\rho^n(\bar{t}, \cdot) \rightharpoonup \rho(\bar{t}, \cdot)$ weak* in L^∞ as $n \to \infty$.

The proof of Point (2) in completely similar to the proof of Point (1) and thus it is omitted. Point (3) follows from the general properties of weak convergence. Finally observe that, passing to the limit (5.67), we get $|\nu|(\Omega) \le 2(L_N - L_0)$. Point (4) is now a consequence of this last inequality. □

5.6. Convergence of the curves $\hat{\gamma}^\varepsilon$

In the previous section we proved the existence of a $(N+4)$-tuple

$$\mathcal{E} := (L_0, \ldots, L_N, \mathtt{x}, \rho, \bar{\rho})$$

where

$$L_0 \le \cdots \le L_N, \qquad \mathtt{x} : [0, T] \times (L_0, L_N] \to \mathbb{R}, \qquad \rho, \bar{\rho} : [0, T] \times (L_0, L_N] \to [-1, 1],$$

such that

a) for every time $t \in [0, T]$, the collection

$$\mathcal{E}(t) := \Big(L_0, \ldots, L_N, \mathtt{x}(t, \cdot), \rho(t, \cdot), \bar{\rho}(t, \cdot)\Big)$$

is an e.o.w.

b) the distributions $D_t\rho$, $D_t\bar{\rho}$ are finite Radon measure on $[0, T] \times \mathbb{R}$.

Therefore, for every fixed time $t \in [0, T]$, with the techniques introduced in Section 5.1, we can construct:

- the sign $\mathcal{S}_k(t)$ of points $x \in \mathbb{R}$ (see (5.2));
- the order relation $\Re(t)$ on $(L_0, L_N]$ (see (5.3));

- the numbers $M_k(t), M(t) \in \mathbb{R}$ (see (5.4));
- the maps $V_k(t), V^(t), \omega_k(t)$ (see (5.5) and (5.7));
- the maps $\hat{\mathtt{x}}_k(t), \hat{\mathtt{x}}(t)$ (see (5.8));
- the curve
$$\hat{\gamma}(t) := \big(\hat{u}(t), \hat{v}_1(t), \dots, \hat{v}_N(t), \sigma_1(t), \dots, \sigma_N(t)\big),$$
and the functions $\hat{f}_k(t) := f_k^{\hat{\gamma}(t)}$, $k = 1, \dots, N$,
$$f_k(t) : [0, M_k(t)] \to \mathbb{R}, \quad f_k(t)(z) := \int_{(0,z]} \big(V_k(t)\big)_\sharp \Big(\bar{\rho}(t)\lambda^{\hat{\gamma}(t)}\mathcal{L}^1|_{(L_{k-1},L_k]}\Big)(d\zeta),$$
such that the fixed point system
$$\begin{cases} \hat{u}(t)(z) := \int_{(0,z]} V(t)_\sharp\big(\rho(t)r^{\hat{\gamma}(t)}\mathcal{L}^1|_{(L_0,L_N]}\big)(d\zeta), \\ \hat{v}_k(t)(z) := \operatorname{sign}\Big(\mathcal{S}_k(t)(\hat{\mathtt{x}}_k(t)(z))\Big)\Big(f_k(t)(z) - \operatorname{conv}_{(\hat{\mathtt{x}}_k(t))^{-1}(\hat{\mathtt{x}}_k(t)(z))} f_k(t)(z)\Big), \ k = 1, \dots, N, \\ \hat{\sigma}_k(z) := \frac{d}{dz}\operatorname{conv}_{(\hat{\mathtt{x}}_k(t))^{-1}(\hat{\mathtt{x}}_k(t)(z))} f_k^{\hat{\gamma}}(z), \qquad k = 1, \dots, N, \end{cases} \tag{5.69}$$
is satisfied (see Proposition 5.19).

As already pointed out at the end of Section 5.2, the main difficulty in proving that $\mathcal{E}$ is a Lagrangian representation consists in showing that the objects
$$\Re,\ M_k,\ M,\ V_k,\ V,\ \hat{\mathtt{x}}_k,\ \hat{\mathtt{x}},\ \hat{u},\ \hat{v}_k,\ \hat{\sigma}_k$$
constructed with the techniques of Section 5.1 starting from $\mathtt{x}, \rho, \bar{\rho}$ are the limits of the corresponding objects
$$\Re^\varepsilon,\ M_k^\varepsilon,\ M^\varepsilon,\ V_k^\varepsilon,\ V^\varepsilon,\ \hat{\mathtt{x}}_k^\varepsilon,\ \hat{\mathtt{x}}^\varepsilon,\ \hat{u}^\varepsilon,\ \hat{v}_k^\varepsilon,\ \hat{\sigma}_k^\varepsilon$$
(see Section 5.4) constructed with the techniques of Section 5.1 starting from $\mathtt{x}^\varepsilon, \rho^\varepsilon, \bar{\rho}^\varepsilon$ (up to subsequence and in the appropriate topologies). This is the aim of this section. In particular we will show that this convergence holds for every time up to a countable set $\mathcal{Z} := \mathcal{Z}_1 \cup \mathcal{Z}_2$, where $\mathcal{Z}_1$ was defined in (5.68) and $\mathcal{Z}_2$ will be defined in (5.70).

5.6.1. Convergence of the interaction measure. We start our analysis with the convergence of the interaction measures μ^n.

PROPOSITION 5.40. *There exists a positive, finite Radon measure μ on $[0,T] \times \mathbb{R}$ and a subsequence of (μ^n), still denoted by (μ^n), such that (μ^n) converges weakly* to μ as $n \to \infty$.*

PROOF. The proof is an immediate consequence of (5.58) and Theorem 1.27. □

Let us define also the *set of times when a strong interaction occurs* as
$$\mathcal{Z}_2 := \bigg\{ t \in [0,T] \ \bigg|\ \mu\Big(\{t\} \times \mathbb{R}\Big) \neq 0 \bigg\}. \tag{5.70}$$
Clearly $\mathcal{Z}_2$ is at most countable. Set $\mathcal{Z} := \mathcal{Z}_1 \cup \mathcal{Z}_2$.

5.6.2. Convergence of $\Re^n$. In this section we study the convergence of the relations $\Re^n$. In particular we will show that for every time $t \notin \mathcal{Z}$, $\chi_{\Re^n} \to H$ in L^1 where H is a function which is $\bar{\rho}(t,\cdot)\bar{\rho}(t,\cdot)\mathcal{L}^2$-a.e. equal to $\chi_\Re$.

We need first the following lemma, which says that not too many waves of different families can have the same position.

LEMMA 5.41. *Let $\bar{t} \notin \mathcal{Z}$. Set*

$$E := \{(w, w') \in \mathcal{W}_k \times \mathcal{W}_h \mid k \neq h \text{ and } \mathtt{x}(\bar{t}, w) = \mathtt{x}(\bar{t}, w')\}. \tag{5.71}$$

Then it holds

$$\iint_E \bar{\rho}(\bar{t}, w)\bar{\rho}(\bar{t}, w')dwdw' = 0.$$

In particular, the previous proposition implies that for every $\bar{x} \in \mathbb{R}$, ρ-almost all the waves located in the point $(\bar{t}, \bar{x})$ belong to the same family.

PROOF. Set

$$\Lambda := \min_{k,h} |\lambda_k^{\max} - \lambda_h^{\min}|.$$

Fix any $\delta, \eta > 0$, with $\eta < \Lambda$. As in Proposition 5.37 set, for every $n \in \mathbb{N}$,

$$E_\delta^n := \{(w, w') \in \mathcal{W}_k \times \mathcal{W}_h \mid k \neq h \text{ and } |\mathtt{x}^n(\bar{t}, w) - \mathtt{x}^n(\bar{t}, w')| < \delta\}.$$

We already know, by Proposition 5.37 that

$$\iint_{E_\delta^n} \bar{\rho}^n(\bar{t}, w)\bar{\rho}^n(\bar{t}, w')dwdw' \ \leq \mathcal{O}(1)\mu^n\left(\left[\bar{t} - \frac{\eta+\delta}{\Delta\Lambda - \eta}, \bar{t} + \frac{\eta+\delta}{\Delta\Lambda - \eta}\right] \times \mathbb{R}\right).$$

Now we want to pass to the limit as $n \to \infty$. Define

$$E_\delta := \{(w, w') \in \mathcal{W}_k \times \mathcal{W}_h \mid k \neq h \text{ and } |\mathtt{x}(\bar{t}, w) - \mathtt{x}(\bar{t}, w')| < \delta\}.$$

Observe that for every $\delta > 0$ (up to a countable set), the set

$$\{(w, w') \in (L_0, L_N]^2 \mid |\mathtt{x}(\bar{t}, w) - \mathtt{x}(\bar{t}, w)| = \delta\}$$

is $\mathcal{L}^2$-negliglible and thus we can apply Lemma 1.21 to get

$$E_\delta^n \to E_\delta \text{ in } L^1\big((L_0, L_N]^2\big).$$

Taking now the lim sup as $n \to \infty$ and using Proposition 1.30 and Remark 5.32, we get

$$\iint_{E_\delta} \bar{\rho}(\bar{t}, w)\bar{\rho}(\bar{t}, w')dwdw' \leq \mu\left(\left[\bar{t} - \frac{\eta+\delta}{\Delta\Lambda - \eta}, \bar{t} + \frac{\eta+\delta}{\Delta\Lambda - \eta}\right] \times \mathbb{R}\right).$$

As $\eta \to 0$ and $\delta \to 0$, we finally get

$$\iint_E \bar{\rho}(\bar{t}, w)\bar{\rho}(\bar{t}, w')dwdw' \leq \mu\Big(\{\bar{t}\} \times \mathbb{R}\Big) = 0,$$

since $\bar{t} \notin \mathcal{Z}$. □

COROLLARY 5.42. *For every time $\bar{t} \notin \mathcal{Z}$ and for every $x \in \mathbb{R}$, there exists at most one family k such that*

$$\int_{\mathtt{x}(\bar{t})^{-1}(x) \cap \mathcal{W}_k} \bar{\rho}(\bar{t}, w)dw = \mathcal{L}^1\Big(\hat{\mathtt{x}}_k(\bar{t})^{-1}(x)\Big) \neq 0.$$

PROPOSITION 5.43. *Let $t \notin \mathcal{Z}$. Then there exists a subsequence of $(\chi_{\mathfrak{R}^n(t)})_n$ (which we still denote by $(\chi_{\mathfrak{R}^n(t)})_n$ which converges in L^1 to a BV function $H(t, w, w')$ such that*

$$\iint_{(L_0, L_N]^2} \big|H(t, w, w') - \chi_{\mathfrak{R}(t)}(w, w')\big|\bar{\rho}(t, w)\bar{\rho}(t, w')dwdw' = 0.$$

Proof. Since we are working at fixed time $t \notin \mathcal{Z}$, we will omit to explicitly indicate the dependence on time of the objects under consideration.

First of all observe that, by Corollary 5.4, the functions $(\chi_{\mathfrak{R}^n})_n$ are uniformly BV and thus, up to subsequences, they converge in L^1 and a.e. to some map $H \in L^1\big((L_0, L_N)^2\big)$. Notice also that

$$\text{for } \mathcal{L}^2\text{-a.e. } (w,w') \in (L_0,L_N]^2, \text{ if } \chi_{\mathfrak{R}^n}(w,w') \nrightarrow \chi_{\mathfrak{R}}(w,w'), \text{ then } (w,w') \in E, \tag{5.72}$$

where E is the set defined in (5.71). Indeed, take any $w, w' \in (L_0, L_N]$. Since $\mathtt{x}^n(\bar{t}, \cdot) \to \mathtt{x}(\bar{t}, \cdot)$ a.e., we can assume that $\mathtt{x}^n(\bar{t}, w) \to \mathtt{x}(\bar{t}, w)$ and $\mathtt{x}^n(\bar{t}, w') \to \mathtt{x}(\bar{t}, w')$. Now assume that $(w,w') \notin E$. This means that either w, w' belong to the same family or they belong to different families and, say, $\mathtt{x}(t,w) < \mathtt{x}(t,w')$. If w, w' belong to the same family, then

$$\chi_{\mathfrak{R}^n}(w,w') = \chi_{\mathfrak{R}}(w,w') \text{ for every } n \in \mathbb{N}.$$

On the other side, if w, w' belong to different families and $\mathtt{x}(\bar{t}, w) < \mathtt{x}(\bar{t}, w')$, then

$$\mathtt{x}^n(\bar{t}, w) < \mathtt{x}^n(\bar{t}, w') \text{ for } n \gg 1$$

and thus

$$\chi_{\mathfrak{R}^n}(w,w') = \chi_{\mathfrak{R}}(w,w') \text{ for } n \gg 1.$$

As a consequence of (5.72), we have that $\chi_{\mathfrak{R}^n} \to \chi_{\mathfrak{R}}$ in L^1 w.r.t the measure $\bar{\rho}(\cdot)\bar{\rho}(\cdot)\mathcal{L}^2$. Hence

$$\begin{aligned} 0 &= \lim_n \iint_{(L_0,L_N]^2} \big|\chi_{\mathfrak{R}^n}(w,w) - \chi_{\mathfrak{R}}(w,w')\big|\bar{\rho}(w)\bar{\rho}(w')dwdw' \\ &= \iint_{(L_0,L_N]^2} \big|H(w,w) - \chi_{\mathfrak{R}}(w,w')\big|\bar{\rho}(w)\bar{\rho}(w')dwdw', \end{aligned}$$

and thus $(\chi_{\mathfrak{R}^n})$ converges in L^1 to a function H which is $\bar{\rho}(\cdot)\bar{\rho}(\cdot)\mathcal{L}^2$-a.e. equal to $\chi_{\mathfrak{R}}$. □

5.6.3. Convergence of $\hat{V}^n$, $\hat{V}^n_k$, $\hat{\mathtt{x}}^n_k$. We prove now that the three functions

$$V^n(t), V^n_k(t), \hat{\mathtt{x}}^n_k(t)$$

converge respectively to $V(t), V_k(t), \hat{\mathtt{x}}(t)$ for every time $t \notin \mathcal{Z}$ in the appropriate sense, as $n \to \infty$. See (5.5) for the definition of V, V_k and (5.8) for the definition of $\hat{\mathtt{x}}_k$. First of all observe that, for every $t \notin \mathcal{Z}$,

$$M^n(t) := \int_{L_0}^{L_N} \bar{\rho}^n(t,w)dw \to \int_{L_0}^{L_N} \bar{\rho}(t,w)dw =: M(t).$$

and similarly,

$$M^n_k(t) \to M_k(t).$$

See (5.4) for the definition of M_k, M.

The convergence of V^n_k is proved in the next lemma.

Proposition 5.44. *For every $t \notin \mathcal{Z}$ and for every k, $V^n_k \to V_k$ uniformly.*

Proof. Observe that for every $w \in \mathcal{W}_k$,

$$V^n_k(t,w) := \int_{L_{k-1}}^{w} \bar{\rho}^n(t,y)dy = \int_{L_{k-1}}^{L_k} \chi_{(L_{k-1},w]}(y)\bar{\rho}^n(t,y)dy$$

and thus, since $\bar{\rho}^n(t)$ weakly* converges to $\bar{\rho}(t)$, $V^n_k(t,w) \to V_k(t,w)$. The convergence is uniform, because the $(V^n_k(t,\cdot))_n$ are uniformly Lipschitz. □

The convergence of V^n is stated in the next lemma. Similarly to what we did for the convergence of the relation $\Re$, also in this lemma we prove that (V^n) converge to a map $\tilde{V}$ which is $\bar{\rho}(t,\cdot)\mathcal{L}^1$-a.e. equal to V.

PROPOSITION 5.45. *Let $t \notin \mathcal{Z}$. Up to subsequences, the maps $V^n(t,\cdot)$ converges in L^1 and a.e. to a map $\tilde{V}(t,\cdot)$ which is $\bar{\rho}\mathcal{L}^1$-a.e. equal to $V(t,\cdot)$.*

PROOF. Since we work at fixed time $t \notin \mathcal{Z}$, we do not explicitly denote the time dependence. Since $V^n|_{(L_{k-1},L_k]}$ is increasing for every k, the family of maps (V^n) is precompact in L^1 and thus it admits a converging subsequence (still denoted by $(V^n)_n$) which tends to some $\tilde{V} \in L^1((L_0,L_N])$. We want to prove that $\tilde{V}(w) = V(w)$ for $\bar{\rho}\mathcal{L}^1$-a.e. $w \in (L_0,L_N]$. By Proposition 5.43, for $\bar{\rho}\mathcal{L}^1$-a.e. w,

$$\chi_{\Re^n}(\cdot,w) \to H(\cdot,w) \text{ in } L^1 \text{ up to subsequences}$$

and

$$H(y,w) = \chi_{\Re}(y,w) \text{ for } \bar{\rho}\mathcal{L}^1\text{-a.e. } w.$$

Therefore

$$V^n(w) = \int_{L_0}^{L_N} \chi_{\Re^n}(y,w)\bar{\rho}^n(y)dy \to \int_{L_0}^{L_N} H(y,w)\bar{\rho}(y)dy = \int_{L_0}^{L_N} \chi_{\Re}(y,w)\bar{\rho}(y)dy = V(w).$$

Hence $V^n \to V$ for $\bar{\rho}\mathcal{L}^1$-a.e. w and thus $\tilde{V} = V$ for $\bar{\rho}\mathcal{L}^1$-a.e. w. □

We conclude this section with the convergence of the maps $\hat{\mathtt{x}}^n_k$.

PROPOSITION 5.46. *Let $t \notin \mathcal{Z}$ be a fixed time. It holds $\hat{\mathtt{x}}^n_k(t,\cdot) \to \hat{\mathtt{x}}_k(t,\cdot)$ in L^1, in the sense that $M^n_k(t) \to M_k(t)$ and*

$$\int_{(0,\min\{M^n_k(t),M_k(t)\}]} \big|\hat{\mathtt{x}}^n_k(t,z) - \hat{\mathtt{x}}_k(t,z)\big|dz \to 0.$$

PROOF. As before, since we work at fixed time $t \notin \mathcal{Z}$, we will omit to denote the explicit dependence on the time. Our aim is to prove that for every subsequence n_j there is a sub-subsequence n_{j_l} such that $\hat{\mathtt{x}}^{n_{j_l}}_k(z) \to \hat{\mathtt{x}}_k(z)$ in L^1. Define first

$$\begin{aligned}\tilde{E} := &\big\{z \in (0,M_k] \text{ such that } V_k^{-1}(z) \text{ is not single-valued}\big\} \\ &\cup \bigcup_{n\in\mathbb{N}} \big\{z \in (0,M_k] \text{ such that } (V^n_k)^{-1}(z) \text{ is not single-valued}\big\}\end{aligned}$$

and

$$E := \big\{z \in (0,M_k] \setminus \tilde{E} \text{ such that } \mathtt{x}(t) \text{ is not continuos at } V_k^{-1}(z)\big\}.$$

Clearly $\mathcal{L}^1(E) = 0$. Let us fix a subsequence of $\hat{\mathtt{x}}^n_k$, which we will denote still by $\hat{\mathtt{x}}^n_k$. Our goal is to find a sub-subsequence converging to $\hat{\mathtt{x}}_k$ in L^1.

Since $\hat{\mathtt{x}}^n_k$ is a family of increasing maps, there exists a subsequence, which we will still denote by n, and an increasing map $\hat{\mathtt{y}}_k : (0,M_k] \to \mathbb{R}$ such that

$$\hat{\mathtt{x}}^n_k = \mathtt{x}^n \circ (V^n_k)^{-1} \to \hat{\mathtt{y}}_k \text{ in } L^1 \text{ and a.e.}$$

If we prove that $\hat{\mathtt{y}}_k = \hat{\mathtt{x}}_k$ a.e., we are done. More precisely, we will prove that for every $\delta > 0$ there is a subset $F_\delta \subseteq (0,M_k]$ such that $\mathcal{L}^1(F_\delta) \le \delta$ and

$$\hat{\mathtt{y}}_k = \hat{\mathtt{x}}_k \text{ on } (0,M_k] \setminus F_\delta. \tag{5.73}$$

Clearly, this is enough.

Fix thus $\delta > 0$. There is a subset $D_\delta \subseteq (L_{k-1},L_k]$ such that $\mathcal{L}^1(D_\delta) \le \delta$ and

$$\mathtt{x}^n \to \mathtt{x} \text{ uniformly on } (L_{k-1},L_k] \setminus D_\delta. \tag{5.74}$$

Define

$$F_\delta := \bigcup_{j\in\mathbb{N}} \bigcap_{n\geq j} V_k^n(D_\delta) \cup E \cup \big\{z \in (0, M_k] \;\big|\; \hat{\mathtt{x}}_k^n \text{ not converges to } \hat{\mathtt{y}}_k\big\}.$$

We have

$$\mathcal{L}^1(F_\delta) = \lim_{j\to\infty} \mathcal{L}^1\Big(\bigcap_{n\geq j} V_k^n(D_\delta)\Big) \leq \limsup_{j\to\infty} \mathcal{L}^1(V_k^j(D_\delta)) \leq \delta,$$

since the maps V_k^j are uniformly Lipschitz, with Lipschitz constant less or equal than 1. We thus have to prove that (5.73) holds. Fix any $z \in (0, M_k] \setminus F_\delta$. By definition, there is a subsequence n_j such that $z \notin V_k^{n_j}(D_\delta)$ for every j, or, in other words, $(V_k^{n_j})^{-1}(z) \notin D_\delta$ for every j. Notice that the subsequence depends both on δ and on z.

$$\begin{aligned}
|\hat{\mathtt{y}}_k(z) - \hat{\mathtt{x}}_k(z)| &\leq |\hat{\mathtt{y}}_k(z) - \hat{\mathtt{x}}_k^{n_j}(z)| + |\hat{\mathtt{x}}_k^{n_j}(z) - \hat{\mathtt{x}}_k(z)| \\
&\leq |\hat{\mathtt{y}}_k(z) - \hat{\mathtt{x}}_k^{n_j}(z)| + |\mathtt{x}^{n_j}((V_k^{n_j})^{-1})(z) - \mathtt{x}(V_k^{-1}(z))| \\
&\leq |\hat{\mathtt{y}}_k(z) - \hat{\mathtt{x}}_k^{n_j}(z)| + |\mathtt{x}^{n_j}((V_k^{n_j})^{-1})(z) - \mathtt{x}((V_k^{n_j})^{-1})(z)| \\
&\quad + |\mathtt{x}((V_k^{n_j})^{-1})(z) - \mathtt{x}(V_k^{-1}(z))| \\
&\leq |\hat{\mathtt{y}}_k(z) - \hat{\mathtt{x}}_k^{n_j}(z)| + \sup_{(L_{k-1},L_k]\setminus D_\delta} |\mathtt{x}^{n_j} - \mathtt{x}| + |\mathtt{x}((V_k^{n_j})^{-1})(z) - \mathtt{x}(V_k^{-1}(z))|.
\end{aligned}$$

As $j \to \infty$,

- the first term tends to 0 by the hypothesis on $\hat{\mathtt{y}}_k$ and the definition of F_δ;
- the second term tends to zero by (5.74);
- the third term tends to zero since $z \notin E$ and thus, by Proposition 5.44, $(V_k^{n_j})^{-1}(z) \to V_k^{-1}(z)$ and $V_k^{-1}(z)$ is a continuity point of $\mathtt{x}$.

We thus get (5.73) on $(0, M_k] \setminus F_\delta$, which is what we wanted to get. $\square$

5.6.4. Compactness of $\hat{u}^n, \hat{v}_k^n, \hat{\sigma}_k^n$. Up to now we have proved the convergence of the interaction measures μ^n, the relations $\Re^n$, the functions V_k^n, V^n and $\hat{\mathtt{x}}_k^n$ at any fixed time $t \notin \mathcal{Z}$. It is thus left to show that the components $\hat{u}^n$, $\hat{v}_k^n$, $\hat{\sigma}_k^n$ of the curve $\hat{\gamma}^n$ converge to the components $\hat{u}$, $\hat{v}_k$, $\hat{\sigma}_k$ of the limit curve $\hat{\gamma}$ at every fixed time $t \notin \mathcal{Z}$. This is done in this section and the next three Sections 5.6.5, 5.6.6, 5.6.7. Since we work at fixed time, we will assume in this and the next three sections that $\bar{t} \notin \mathcal{Z}$ is a fixed time and we will not anymore indicate the explicit time dependence.

The technique we will adopt is the following. First of all we will prove that the sequences (u^n), (v_k^n), (σ_k^n) are pre-compact in the appropriate topology and thus, up to subsequences, $\hat{u}^n \to \tilde{u}$, $\hat{v}_k^n \to \tilde{v}_k$, $\hat{\sigma}_k^n \to \tilde{\sigma}_k$ (see Proposition 5.47 below) Then we will prove that $(\tilde{u}, \tilde{v}_1, \dots, \tilde{v}_N, \tilde{\sigma}_1, \dots, \tilde{\sigma}_N)$ satisfy the system (5.69). Since, by Proposition 5.19, $(\hat{u}, \hat{v}_1, \dots, \hat{v}_N, \hat{\sigma}_1, \dots, \hat{\sigma}_N)$ is the unique solution of (5.69), we get that $\hat{u}^n \to \hat{u}$, $\hat{v}_k^n \to \hat{v}_k$ and $\hat{\sigma}_k^n \to \hat{\sigma}_k$.

We first prove that the sequences (u^n), (v_k^n), (σ_k^n) are pre-compact in the appropriate topology.

PROPOSITION 5.47. *There exist maps*

$$\begin{aligned}
&\tilde{u} : [0, M] \to \mathbb{R}^n, \\
&\tilde{v}_k : [0, M_k] \to \mathbb{R}, \qquad k = 1, \dots, N, \\
&\tilde{\sigma}_k : [0, M_k] \to \mathbb{R}^n, \qquad k = 1, \dots, N, \\
&\tilde{f}_k : [0, M_k] \to \mathbb{R}, \qquad k = 1, \dots, N,
\end{aligned}$$

such that for every $k=1,\dots,N$, $\tilde{u},\tilde{v}_k,\tilde{f}_k$ are Lipschitz, $\tilde{\sigma}_k$ are BV, and as $n\to\infty$, up to subsequences,

- $\hat{u}^n \to \tilde{u}$ *uniformly, i.e.*
$$M^n \to M \text{ and } \sup_{z\in[0,\min\{M^n,M\}]} |\hat{u}^n(z)-\tilde{u}(z)| \to 0;$$
- $\hat{v}_k^n \to \tilde{v}_k$ *uniformly, i.e.*
$$M_k^n \to M_k \text{ and } \sup_{z\in[0,\min\{M_k^n,M_k\}]} |\hat{u}^n(z)-\tilde{u}(z)| \to 0;$$
- $\hat{\sigma}_k^n \to \sigma_k$ *in L^1 (and a.e.) i.e.*
$$\int_0^{\min\{M_k^n,M_k\}} |\hat{\sigma}_k^n(z)-\tilde{\sigma}_k(z)|dz \to 0 \text{ and for a.e. } z\in[0,M_k],\ \sigma_k^n(z)\to\tilde{\sigma}_k(z);$$
- $\hat{D}f_k^n \to \tilde{D}f_k$ *in L^1 (and a.e.) i.e.*
$$\int_0^{\min\{M_k^n,M_k\}} |D\hat{f}_k^n(z)-D\tilde{f}_k(z)|dz \to 0 \text{ and for a.e. } z\in[0,M_k],\ Df_k^n(z)\to D\tilde{f}_k(z);$$
- $f_k^n \to \tilde{f}_k$ *uniformly, i.e.*
$$\sup_{z\in[0,\min\{M_k^n,M_k\}]} |\hat{f}_k^n(z)-\tilde{f}_k(z)| \to 0,$$

where we set, for simplicity, $f_k^n = f_k^{\hat{\gamma}^n}$.

Proof. The compactness of the families $(\hat{u}^n)_n,(\hat{v}_k^n)_n,(\hat{\sigma}_k^n)_n$ is an easy consequence of Proposition 5.19 e Lemma 5.18. The compactness of the family $(D\hat{f}_k^n)_n$ is an easy consequence of Lemma 5.18. The convergence of the family $(\hat{f}_k^n)_n$ follows from the convergence of their derivatives. □

5.6.5. Analysis on $\tilde{u}$ and $\tilde{f}_k$. In this section we prove that the map $\tilde{u}$ obtained as limit of the sequence $(\hat{u}^n)$ in Proposition 5.47 satisfies the first equation in the system (5.69) and that $\tilde{f}_k$ obtained as limit of the sequence $(\hat{f}^n)$ in Proposition 5.47 is exactly the flux associated to the $(2N+1)$-tuple $(\tilde{u},\tilde{v}_1,\dots,\tilde{v}_N,\tilde{\sigma}_1,\dots,\tilde{\sigma}_N)$. We need first the following three lemmas.

Lemma 5.48. *It holds*
$$\int_{L_0}^{L_N} \Big|\hat{u}^n\big(V^n(w)\big)-\tilde{u}\big(V(w)\big)\Big|\big|\rho^n(w)\big|dw \to 0 \text{ as } n\to\infty.$$

Proof. Using Proposition 5.45 and the fact that the maps $(\hat{u}^n)_n$ are uniformly Lipschitz, we have that
$$\int_{L_0}^{L_N} \Big|\hat{u}^n\big(V^n(w)\big)-\tilde{u}\big(V(w)\big)\Big|\big|\rho^n(w)\big|dw \to \int_{L_0}^{L_N} \Big|\hat{u}\big(\tilde{V}(w)\big)-\tilde{u}\big(V(w)\big)\Big|\big|\rho(w)\big|dw = 0,$$
which is what we wanted to prove. □

Lemma 5.49. *For every $k=1,\dots,N$, the map $\hat{v}_k^n\circ V_k^n$ converges to $\tilde{v}_k\circ V_k$ uniformly.*

PROOF. We have

$$\begin{aligned}|\hat{v}_k^n(V_k^n(w)) - \tilde{v}_k(V_k(w))| &\\ &\leq |\hat{v}_k^n(V_k^n(w)) - \tilde{v}_k(V_k^n(w))| + |\tilde{v}_k(V_k^n(w)) - \tilde{v}_k(V_k(w))| \\ &\leq \sup_{[0,\min\{M_k,M_k^n\}]} |\hat{v}_k^n(z) - \tilde{v}_k(z)| + \mathrm{Lip}(\tilde{v}_k) \sup_{(L_{k-1},L_k]} |V_k^n(w) - V_k(w)|,\end{aligned}$$

and thus

$$\begin{aligned}\sup_{(L_{k-1},L_k]} &|\hat{v}_k^n(V_k^n(w)) - \tilde{v}_k(V_k(w))| \\ &\leq \sup_{[0,\min\{M_k,M_k^n\}]} |\hat{v}_k^n(z) - \tilde{v}_k(z)| + \mathrm{Lip}(\tilde{v}_k) \sup_{(L_{k-1},L_k]} |V_k^n(w) - V_k(w)|.\end{aligned}$$

The first term tends to zero by Proposition 5.47, while the second term tends to zero by Proposition 5.44. □

LEMMA 5.50. *For every $k = 1, \dots, N$, it holds*

$$\int_{L_{k-1}}^{L_k} \Big|\hat{\sigma}_k^n\big(V_k^n(w)\big) - \tilde{\sigma}_k\big(V_k(w)\big)\Big|\Big|\rho^n(w)\Big|dw \to 0 \text{ as } n \to \infty.$$

PROOF. To avoid too heavy notations, we assume for simplicity that $M_k^n = M_k$ for every $n \in \mathbb{N}$. In this way, $\tilde{\sigma}_k$ and all the $\hat{\sigma}_k^n$, $n \in \mathbb{N}$, have the same domain. The general case follows from the fact that $M_k^n \to M_k$ as $n \to \infty$ and Corollary 5.7. We prove that for every subsequence, there exists a sub-subsequence which converges to 0. Let us thus extract any subsequence, which we still denote with the same index. We have

$$\begin{aligned}&\int_{L_{k-1}}^{L_k} \Big|\hat{\sigma}_k^n\big(V_k^n(w)\big) - \tilde{\sigma}_k\big(V_k(w)\big)\Big|\Big|\rho^n(w)\Big|dw \\ &\quad\leq \int_{L_{k-1}}^{L_k} \Big|\hat{\sigma}_k^n\big(V_k^n(w)\big) - \tilde{\sigma}_k\big(V_k^n(w)\big)\Big|\Big|\rho^n(w)\Big|dw + \int_{L_{k-1}}^{L_k} \Big|\tilde{\sigma}_k\big(V_k^n(w)\big) - \tilde{\sigma}_k\big(V_k(w)\big)\Big|\Big|\rho^n(w)\Big|dw \\ &\quad\text{(making the change of variable } z = V_k(w)) \\ &\quad= \int_0^{M_k} \big|\hat{\sigma}_k^n(z) - \tilde{\sigma}_k(z)\big|dz + \int_{L_{k-1}}^{L_k} \Big|\tilde{\sigma}_k\big(V_k^n(w)\big) - \tilde{\sigma}_k\big(V_k(w)\big)\Big|\Big|\rho^n(w)\Big|dw.\end{aligned}$$

The first term tends to zero as $n \to \infty$ by Proposition 5.47. Now observe that, since V_k^n and V_k are increasing and $\tilde{\sigma}_k$ is BV, we get

$$\text{e.Tot.Var.}\Big(\tilde{\sigma}_k \circ V_k^n; (L_{k-1}, L_k)\Big) \leq \text{e.Tot.Var.}\Big(\tilde{\sigma}_k; (0, M_k)\Big) \text{ for every } n \in \mathbb{N}.$$

Therefore there exists a map $J : [L_{k-1}, L_k] \to \mathbb{R}$ and a subsequence j_n such that $\tilde{\sigma}_k \circ V_k^{n_j} \to J$ as $j \to \infty$ in L^1. Since $|\rho^n| \overset{\star}{\rightharpoonup} \bar{\rho}$ weakly* in L^∞, we get that

$$\int_{L_{k-1}}^{L_k} \Big|\tilde{\sigma}_k\big(V_k^{n_j}(w)\big) - \tilde{\sigma}_k\big(V_k(w)\big)\Big|\Big|\rho^n(w)\Big|dw \to \int_{L_{k-1}}^{L_k} \Big|J(w) - \tilde{\sigma}_k\big(V_k(w)\big)\Big|\bar{\rho}(w)dw.$$

If we prove that $J(w) = \tilde{\sigma}_k(V_k(w))$ for $\bar{\rho}\mathcal{L}^1$-a.e. $w \in (L_{k-1}, L_k]$, the proof is concluded. Define

$$F := \big\{z \in [0, M_k] \;\big|\; \tilde{\sigma}_k \text{ is not continuous in } z\big\}.$$

Since $\tilde{\sigma}_k$ is BV, $\mathcal{L}^1(F) = 0$. Define also

$$E := V_k^{-1}(F) \cup \big\{w \in (L_{k-1}, L_k] \;\big|\; V_k^n(w) \text{ does not converge to } V_k(w)\big\}.$$

By Corollary 5.7 and Proposition 5.44, $\bar{\rho}\mathcal{L}^1(E) = 0$. Now notice that if $w \notin E<$ then $V_k^n(w) \to V_k(w)$ and $V_k(w)$ is a continuity point of $\tilde{\sigma}_k$. Therefore $\tilde{\sigma}_k(V_k^n(w)) \to \tilde{\sigma}_k(V_k(w))$ and thus $J(w) = \tilde{\sigma}_k(V_k(w))$, which is what we wanted to get. □

PROPOSITION 5.51 (Convergence of $\hat{u}$). *For every $z \in [0, M]$, it holds*

$$\tilde{u}(z) = \int_{(0,z]} V_\sharp\Big(\rho(\bar{t})\tilde{r}^{\tilde{\gamma}}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta). \tag{5.75}$$

PROOF. It is enough to show that

$$\int_{(0,z]} V_\sharp^n\Big(\rho^n(\bar{t})r^{\hat{\gamma}^n}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) \to \int_{(0,z]} V_\sharp\Big(\rho(\bar{t})\tilde{r}^{\tilde{\gamma}}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) \text{ as } n \to \infty$$

for a.e. $z \in [0, M]$, since both the l.h.s. and the r.h.s. of (5.77) are continuous functions of z We have

$$\begin{aligned}
&\bigg|\int_{(0,z]} V_\sharp^n\Big(\rho^n(\bar{t})r^{\hat{\gamma}^n}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta) - \int_{(0,z]} V_\sharp\Big(\rho(\bar{t})\tilde{r}^{\tilde{\gamma}}\mathcal{L}^1|_{(L_0,L_N]}\Big)(d\zeta)\bigg| \\
&\leq \sum_{k=1}^N \bigg|\int_{\substack{(V^n)^{-1}((0,z]) \\ \cap(L_{k-1},L_k]}} \rho^n(\bar{t},w)\tilde{r}_k\big(\hat{u}^n(V^n(w)), \hat{v}_k^n(V_k^n(w)), \hat{\sigma}_k^n(V_k^n(w))\big)dw \\
&\qquad\qquad - \int_{\substack{V^{-1}((0,z]) \\ \cap(L_{k-1},L_k]}} \rho(\bar{t},w)\tilde{r}_k\big(\tilde{u}(V(w)), \tilde{v}_k(V_k(w)), \tilde{\sigma}_k(V_k(w))\big)dw\bigg| \\
&\leq \sum_{k=1}^N \int_{L_{k-1}}^{L_k} \big|\rho^n\big|\Big|\tilde{r}_k\big(\hat{u}^n(V^n), \hat{v}_k^n(V_k^n), \hat{\sigma}_k^n(V_k^n)\big) - \tilde{r}_k\big(\tilde{u}(V), \tilde{v}_k(V_k), \tilde{\sigma}_k(V_k)\big)\Big|dw \\
&\qquad + \bigg|\int_{L_{k-1}}^{L_k} \Big(\rho^n\chi_{(V^n)^{-1}((0,z])} - \rho^n\chi_{V^{-1}((0,z])}\Big)\tilde{r}_k\big(\tilde{u}(V), \tilde{v}_k(V_k), \tilde{\sigma}_k(V_k)\big)dw\bigg| \\
&\qquad + \bigg|\int_{L_{k-1}}^{L_k} \Big(\rho^n\chi_{V^{-1}((0,z])} - \rho\chi_{V^{-1}((0,z])}\Big)\tilde{r}_k\big(\tilde{u}(V), \tilde{v}_k(V_k), \tilde{\sigma}_k(V_k)\big)dw\bigg|.
\end{aligned}$$

We now separately study the three terms in the r.h.s. of the last inequality.

The first term can be estimated as follows.

$$\begin{aligned}
&\int_{L_{k-1}}^{L_k} \big|\rho^n\big|\Big|\tilde{r}_k\big(\hat{u}^n(V^n), \hat{v}_k^n(V_k^n), \hat{\sigma}_k^n(V_k^n)\big) - \tilde{r}_k\big(\tilde{u}(V), \tilde{v}_k(V_k), \tilde{\sigma}_k(V_k)\big)\Big|dw \\
&\qquad \leq \mathcal{O}(1)\bigg\{\int_{L_0}^{L_N} \big|\hat{u}^n(V^n) - \tilde{u}(V)\big||\rho^n|dw + \big\|\hat{v}_k^n(V_k^n) - \tilde{v}_k(V_k)\big\|_\infty \\
&\qquad\qquad + \int_{L_{k-1}}^{L_k} \big|\hat{\sigma}_k^n(V_k^n) - \tilde{\sigma}_k(V_k)\big||\rho^n|dw\bigg\}
\end{aligned}$$

and thus, by Lemmas 5.48, 5.49, 5.50, it tends to zero as $n \to \infty$.

The second term

$$\bigg|\int_{L_{k-1}}^{L_k} \Big(\rho^n\chi_{(V^n)^{-1}((0,z])} - \rho^n\chi_{V^{-1}((0,z])}\Big)\tilde{r}_k\big(\tilde{u}(V), \tilde{v}_k(V_k), \tilde{\sigma}_k(V_k)\big)dw\bigg| \tag{5.76}$$

tends to zero as $n \to \infty$ for a.e. $z \in (0, M]$. Indeed notice first that since since V is increasing on each $(L_{k-1}, L_k]$, then $\mathcal{L}^1(V^{-1}(z)) = 0$ for a.e. $z \in [0, M]$. Therefore, since $V^n \to V$ a.e. and using Lemma 1.21, we get that $\chi_{(V^n)^{-1}((0,z]} \to V^{-1}((0, z])$ in $L^1((L_0, L_N])$

for a.e. $z \in [0, M]$. Using now again the weak* convergence of ρ^n to ρ, we get that for a.e. $z \in [0, M]$, the term in (5.76) tends to zero as $n \to \infty$.

The third term

$$\left| \int_{L_{k-1}}^{L_k} \Big(\rho^n \chi_{V^{-1}((0,z])} - \rho \chi_{V^{-1}((0,z])} \Big) \tilde{r}_k \big(\tilde{u}(V), \tilde{v}_k(V_k), \tilde{\sigma}_k(V_k) \big) dw \right|$$

tends to zero as $n \to \infty$ since ρ^n weakly* converges to ρ in L^∞.

We thus have that for a.e. $z \in [0, M]$, equality (5.77) holds. Since both the l.h.s. and the r.h.s. of (5.77) are continuous functions of z, the equality holds for every $z \in [0, M]$. □

PROPOSITION 5.52 (Convergence of $\hat{f}_k^n$). *For every $k = 1, \dots, N$ and for every $z \in [0, M_k]$, it holds*

$$\tilde{f}_k(z) = \int_{(0,z]} (V_k)_\sharp \Big(\bar{\rho}(\bar{t}) \tilde{\lambda}^{\tilde{\gamma}} \mathcal{L}^1|_{(L_{k-1}, L_k]} \Big) (d\zeta). \tag{5.77}$$

The proof is completely similar to the proof of Proposition 5.51 and thus it is omitted.

5.6.6. Analysis of $\tilde{\sigma}_k$. In this section we prove that the map $\tilde{\sigma}$ obtained as limit of the sequence $(\hat{\sigma}^n)$ in Proposition 5.47 satisfies the third equation in the system (5.69). Here the analysis is not so easy as in the previous section. We need first to prove that all the waves which have the same position have also the same speed. Then, using this fact, we prove that $\tilde{\sigma}_k$ satisfies the third equation in the system (5.69).

Let us start with the proof of the fact that all the waves which have the same position have also the same speed. The proof of this property is based on the fact the the interaction measures μ^n bounds the change in speed of the waves (see Proposition 5.34) and thus, in the limit, if $\bar{t}$ is a time when a strong change of speed occurs, it must hold $\mu(\{\bar{t}\} \times \mathbb{R}) > 0$, i.e. $\bar{t} \in \mathcal{Z}$, a contradiction since we are assuming that $\bar{t} \notin \mathcal{Z}$. The two next lemmas require the analysis of the solution u not only at time $\bar{t}$, but also at time t in a neighborhood of $\bar{t}$. Therefore, in the two next lemmas, we will explicitly write the time dependence.

LEMMA 5.53. *Let $t \neq \bar{t}$, $\bar{t} \notin \mathcal{Z}$. For every $k = 1, \dots, N$ and for every $x \in \mathbb{R}$, setting $I := \big[\min\{\bar{t}, t\}, \max\{\bar{t}, t\}\big]$, it holds*

$$\int_{(L_{k-1}, L_k] \cap \mathtt{x}(\bar{t})^{-1}(x)} \left| \frac{\mathtt{x}(t, w) - \mathtt{x}(\bar{t}, w)}{t - \bar{t}} - \hat{\sigma}_k(\bar{t}, V_k(t, w)) \right| \bar{\rho}(\bar{t}, w) dw \le 4C\mu(I \times \mathbb{R}),$$

where C is the constant which appears in (2.15).

PROOF. Let us prove the lemma only in the case $t > \bar{t}$, the other case being completely similar. Fix $\eta > 0$. Let $\bar{\varepsilon}$ be given by Lemma 5.35. Fix any $n \gg 1$ such that $\varepsilon_n \le \bar{\varepsilon}$. Take any time t such that $t - \bar{t} > \eta$. For every $w \in (L_{k-1}, L_k]$, define the auxiliary map

$$\mathtt{y}^n(\tau, w) := \mathtt{x}^n(\bar{t}, w) + \bar{\sigma}^n(\bar{t}, w)(\tau - \bar{t}).$$

Define also the sets

$$\begin{aligned} E_k^n &:= \big\{ w \in (L_{k-1}, L_k] \ \big|\ |\rho^n(\bar{t}, w)| = 1 \big\}, \\ A_k^n &:= \big\{ w \in E_k^n \ \big|\ |\rho^n(\tau, w)| = 1 \text{ for every } \tau \in [\bar{t}, t] \big\}, \\ B_k^n &:= E_k^n \setminus A_k^n. \end{aligned}$$

Notice that, by Proposition 5.33,

$$\mathcal{L}^1\big(B_k^n\big) \le \mu^n\big([\bar{t}, t] \times \mathbb{R}\big). \tag{5.78}$$

where $\mathcal{I} := \{i \in \mathbb{N} \mid i\varepsilon_n \in I\}$, while, by Proposition 5.34,

$$\begin{aligned}
&\int_{A_k^n} \Big(\max_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w) - \min_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w)\Big)dw \\
&\qquad = \int_{A_k^n} \Big(\max_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w) - \min_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w)\Big)\bar\rho^n(\bar t,w)dw \\
&\qquad \le \mu^n\big([\bar t,t]\times\mathbb{R}\big).
\end{aligned} \tag{5.79}$$

We thus have, by Lemma 5.35,

$$\begin{aligned}
&|\mathtt{x}^n(t,w) - \mathtt{y}^n(t,w)| \\
&\quad \le \left|\mathtt{x}^n(t,w) - \mathtt{x}^n(\bar t,w) - \int_{\bar t}^t \bar\sigma^n(\tau,w)d\tau\right| + \left|\int_{\bar t}^t \bar\sigma^n(\tau,w)d\tau - \bar\sigma^n(\bar t,w)(t-\bar t)\right| \\
&\quad \le 2C(t-\bar t)\left[\eta + \Big(\max_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w) - \min_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w)\Big)\right] + \int_{\bar t}^t \big|\bar\sigma^n(\tau,w) - \bar\sigma^n(\bar t,w)\big|d\tau \\
&\quad \le 4C(t-\bar t)\left[\eta + \Big(\max_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w) - \min_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w)\Big)\right].
\end{aligned}$$

Integrating over all waves in E_k^n we get

$$\begin{aligned}
&\int_{E_k^n} |\mathtt{x}^n(t,w) - \mathtt{y}^n(t,w)|dw \\
&\qquad\le 4C(t-\bar t)\left[\eta \text{Tot.Var.}(\bar u) + \int_{E_k^n}\Big(\max_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w) - \min_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w)\Big)dw\right] \\
&\qquad= 4C(t-\bar t)\left[\eta \text{Tot.Var.}(\bar u) + \int_{A_k^n}\Big(\max_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w) - \min_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w)\Big)dw\right. \\
&\qquad\qquad\left. + \int_{B_k^n}\Big(\max_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w) - \min_{\tau\in[\bar t,t]} \bar\sigma^n(\tau,w)\Big)dw\right] \\
&\qquad\text{(by (5.78) and (5.79))} \\
&\qquad\le 4C(t-\bar t)\Big[\eta \text{Tot.Var.}(\bar u) + \mu^n\big([\bar t,t]\times\mathbb{R}\big)\Big].
\end{aligned}$$

By the definition of E_k^n, we thus get that

$$\int_{L_{k-1}}^{L_k} |\mathtt{x}^n(t,w) - \mathtt{y}^n(t,w)||\rho^n(\bar t,w)|dw \le 4C(t-\bar t)\Big[\eta \text{Tot.Var.}(\bar u) + \mu^n\Big([\bar t,t]\times\mathbb{R}\Big)\Big]. \tag{5.80}$$

We want now to pass to the limit the relation (5.80) as $n\to\infty$. By Propositions 5.38 and 5.39, Lemma 5.50 and using Proposition 1.30 and Remark 5.32, we get

$$\begin{aligned}
&\int_{L_{k-1}}^{L_k} \Big|\mathtt{x}(t,w) - \big(\mathtt{x}(\bar t,w) + \sigma(\bar t,w)(t-\bar t)\big)\Big|\bar\rho(\bar t,w)dw \\
&\qquad \le 4C(t-\bar t)\Big[\eta \text{Tot.Var.}(\bar u) + \mu\Big([\bar t,t]\times\mathbb{R}\Big)\Big]
\end{aligned} \tag{5.81}$$

and (5.81) holds for every $\eta > 0$ and for every t such that $t - \bar t \ge \eta$. We have set, for simplicity,

$$\sigma(\bar t,w) := \hat\sigma_k(\bar t, V_k(\bar t,w))$$

for $w \in \mathcal{W}_k$. Letting $\eta \to 0$, we get

$$\int_{L_{k-1}}^{L_k} \Big|\mathtt{x}(t,w) - \big(\mathtt{x}(\bar{t},w) + \sigma(\bar{t},w)(t-\bar{t})\big)\Big|\bar{\rho}(\bar{t},w)dw \leq 4C(t-\bar{t})\mu\Big([\bar{t},t]\times\mathbb{R}\Big) \tag{5.82}$$

for every $t \geq \bar{t}$. Hence, for every $x \in \mathbb{R}$, we have

$$\int_{(L_{k-1},L_k]\cap\mathtt{x}(\bar{t})^{-1}(x)} \Big|\mathtt{x}(t,w) - x - \sigma(\bar{t},w)(t-\bar{t})\Big|\bar{\rho}(\bar{t},w)dw \leq 4C(t-\bar{t})\mu\Big([\bar{t},t]\times\mathbb{R}\Big) \tag{5.83}$$

and thus

$$\int_{(L_{k-1},L_k]\cap\mathtt{x}(\bar{t})^{-1}(x)} \left|\frac{\mathtt{x}(t,w)-\mathtt{x}(\bar{t},w)}{t-\bar{t}} - \sigma(\bar{t},w)\right|\bar{\rho}(\bar{t},w)dw \leq 4C\mu\Big([\bar{t},t]\times\mathbb{R}\Big),$$

which is what we wanted to get. □

LEMMA 5.54. *Let $\bar{t} \notin \mathcal{Z}$. For every $k = 1,\dots,N$ and for every for every $x \in \mathbb{R}$, there exists a constant σ^* such that*

$$\int_{\mathtt{x}(\bar{t})^{-1}(x)\cap\mathcal{W}_k} |\hat{\sigma}_k(\bar{t},V_k(t,w)) - \sigma^*|\bar{\rho}(\bar{t},w)dw = 0.$$

PROOF. As before set

$$\sigma(\bar{t},w) := \hat{\sigma}_k(\bar{t},V_k(\bar{t},w))$$

for $w \in \mathcal{W}_k$. By Lemma 5.41, we can assume w.l.o.g. that there is one and only one family k such that

$$\int_{(L_{k-1},L_k]\cap\mathtt{x}(\bar{t})^{-1}(x)} \bar{\rho}(w)dw \neq 0.$$

We also know from Lemma 5.53 that for every $t > \bar{t}$, it holds

$$\int_{(L_{k-1},L_k]\cap\mathtt{x}(\bar{t})^{-1}(x)} \left|\frac{\mathtt{x}(t,w)-x}{t-\bar{t}} - \sigma(\bar{t},w)\right|\bar{\rho}(\bar{t},w)dw \leq 4C\mu\Big([\bar{t},t]\times\mathbb{R}\Big)$$

Since the map $w \mapsto (\mathtt{x}(t,w)-x)/(t-\bar{t})$ is increasing, we have, by Proposition 1.42,

$$\begin{aligned}\min\left\{\big\|g-\sigma(\bar{t},\cdot)\big\|_1 \;\middle|\; g \in L^1\Big((L_{k-1},L_k]\cap\mathtt{x}(\bar{t})^{-1}(x);\bar{\rho}\mathcal{L}^1\Big),\ g \text{ monotone increasing}\right\}& \\ \leq \int_{(L_{k-1},L_k]\cap\mathtt{x}(\bar{t})^{-1}(x)} \left|\frac{\mathtt{x}(t,w)-x}{t-\bar{t}} - \sigma(\bar{t},w)\right|\bar{\rho}(\bar{t},w)dw& \\ \leq 4C\mu\Big([\bar{t},t]\times\mathbb{R}\Big).&\end{aligned}$$

Notice that the minimum is taken over the set of functions g which are monotone increasing w.r.t. the measure $\bar{\rho}(\bar{t},\cdot)\mathcal{L}^1$. Letting $t \to \bar{t}$, since $\bar{t} \notin \mathcal{Z}$, we get

$$\min\left\{\big\|g-\sigma(\bar{t},\cdot)\big\|_1 \;\middle|\; g \in L^1\Big((L_{k-1},L_k]\cap\mathtt{x}(\bar{t})^{-1}(x)\Big),\ g \text{ monotone increasing}\right\} = 0$$

and thus $(L_{k-1},L_k]\cap\mathtt{x}(\bar{t})^{-1}(x) \ni w \mapsto \sigma(\bar{t},w)$ is monotone increasing w.r.t the measure $\bar{\rho}\mathcal{L}^1$. A similar argument for $t < \bar{t}$ yields $\sigma(\bar{t},\cdot)$ monotone decreasing w.r.t the measure $\bar{\rho}\mathcal{L}^1$ and thus $\sigma(\bar{t},\cdot)$ must be constant w.r.t the measure $\bar{\rho}\mathcal{L}^1$. □

COROLLARY 5.55. *For every time $\bar{t} \notin \mathcal{Z}$, for every $k = 1,\dots,N$ and for every $x \in \mathbb{R}$, there exists a constant σ^* such that for every $k = 1,\dots,N$,*

$$\int_{\hat{\mathtt{x}}_k(t)^{-1}(x)} |\tilde{\sigma}_k(t,z) - \sigma^*|dz = 0. \tag{5.84}$$

We can finally prove that $\tilde{\sigma}_k$ obtained as limit of the sequence $(\hat{\sigma}_k^n)$ in Proposition 5.47 satisfies the third equation in the system (5.69). From now on the analysis is again at fixed time $\bar{t} \notin \mathcal{Z}$ and thus we will not write the explicit time dependence.

Proposition 5.56. *For a.e.* $z \in [0, M_k]$*, it holds*

$$\tilde{\sigma}_k(z) = D \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} \tilde{f}_k(z).$$

Proof. Set $x := \hat{\mathtt{x}}_k(z)$. Assume first that $\operatorname{card} \hat{\mathtt{x}}_k(\bar{t})^{-1}(x) = 1$. We can always assume that

$$\frac{d}{dz}\hat{f}_k^n(z) \to \frac{d}{dz}\tilde{f}_k(z), \quad \hat{\sigma}_k^n(z) \to \tilde{\sigma}_k(z), \quad \hat{\mathtt{x}}_k^n(z) \to \hat{\mathtt{x}}_k(z), \tag{5.85}$$

since the set of points where the conditions in (5.85) are not satisfied is $\mathcal{L}^1$-negligible. Define $I^n := (\hat{\mathtt{x}}_k^n)^{-1}\big(\hat{\mathtt{x}}_k^n(z)\big)$. By Lemma 1.22, it holds

$$\mathcal{L}^1\big(I^n\big) \to 0 \text{ as } n \to \infty.$$

Notice that, since $\hat{f}_k^n$ is $C^{1,1}$ on I^n (and thus, by Theorem 1.3 also $\operatorname{conv}_{I^n} \hat{f}_k^n$ is $C^{1,1}$ on I^n),

$$\left|\frac{d}{dz}\hat{f}_k^n(z) - \hat{\sigma}_k^n(z)\right| = \left|\frac{d}{dz}\hat{f}_k^n(z) - \frac{d}{dz}\operatorname*{conv}_{I^n} f_k^n(z)\right| \le \mathcal{O}(1)\mathcal{L}^1(I^n).$$

Passing to the limit as $n \to \infty$ and using (5.85), we get $\tilde{\sigma}_k(z) = d\tilde{f}_k(z)/dz$.
Let us now consider the case $\operatorname{card} \hat{\mathtt{x}}_k(\bar{t})^{-1}(x) > 1$, i.e. $\hat{\mathtt{x}}_k(\bar{t})^{-1}(x)$ is an interval. Fix $\delta > 0$, fix $n \in \mathbb{N}$, and assume that $\mathcal{L}^1(\hat{\mathtt{x}}_k^{-1}(\{x-\delta, x+\delta\})) = 0$. This happens for a.e. $\delta > 0$. Define

$$I_\delta := \hat{\mathtt{x}}_k^{-1}\Big(\big[x-\delta, x+\delta\big]\Big), \qquad I_\delta^n := \big(\hat{\mathtt{x}}_k^n\big)^{-1}\Big(\big[x-\delta, x+\delta\big]\Big).$$

Observe first that for a.e. $z \in I_\delta$, there is $\bar{n}$ such that for every $n \ge \bar{n}$, $z \in I_\delta^n$ and

$$\operatorname*{conv}_{I_\delta^n} \hat{f}_k^n(z) \to \operatorname*{conv}_{I_\delta} \tilde{f}_k(z). \tag{5.86}$$

Indeed we can use Lemma 1.21 and the fact that

$$\begin{aligned}
\big|\operatorname*{conv}_{I_\delta^n} \hat{f}_k^n(z) - \operatorname*{conv}_{I_\delta} \tilde{f}_k(z)\big| &\le \big|\operatorname*{conv}_{I_\delta^n} \hat{f}_k^n(z) - \operatorname*{conv}_{I_\delta^n} \tilde{f}_k(z)\big| + \big|\operatorname*{conv}_{I_\delta^n} \tilde{f}_k(z) - \operatorname*{conv}_{I_\delta} \tilde{f}_k(z)\big| \\
\text{(by Proposition 1.11)} &\le \|\hat{f}_k^n - \tilde{f}_k\|_\infty + \mathcal{L}^1\Big(I_\delta \bigtriangleup I_\delta^n\Big),
\end{aligned}$$

and the latter tends to zero by 5.47 and again Lemma 1.21.
Notice now that I_δ^n can be written as union $I_\delta^n = A_1 \cup \dots \cup A_P$, of maximal intervals such that $\hat{\mathtt{x}}_k^n$ is constant on each of these intervals and $\operatorname{conv}_{A_p} \hat{f}_k^n(z) = \int \hat{\sigma}_k^n(z)$ for every $z \in A_p$. Therefore we have

$$\operatorname*{conv}_{I_\delta^n} \hat{f}_k^n(z) \le \bigcup_{p=1}^P \operatorname*{conv}_{A_p} \hat{f}_k^n(z) = \int_0^z \hat{\sigma}_k^n(\zeta)d\zeta \le \hat{f}_k^n(z), \quad \text{for a.e. } z \in I_\delta^n.$$

Passing to the limit, using (5.86) and Proposition 5.47, we get

$$\operatorname*{conv}_{I_\delta} \tilde{f}_k(z) \le \int_0^z \tilde{\sigma}_k(\zeta)d\zeta \le \tilde{f}_k(z) \quad \text{for a.e. } z \in I_\delta.$$

and passing to the limit as $\delta \to 0$,

$$\operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(x)} \tilde{f}_k(z) \le \int_0^z \tilde{\sigma}_k(\zeta)d\zeta \le \tilde{f}_k(z) \quad \text{for a.e. } z \in \hat{\mathtt{x}}_k^{-1}(x). \tag{5.87}$$

We already know by Lemma 5.54 that $\tilde{\sigma}_k$ is equal to some constant σ^* on $\hat{\mathtt{x}}_k^{-1}(x)$ and thus from (5.87) we get

$$\operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(x)} \tilde{f}_k(z) \leq a + \sigma^* \cdot \big(z - \inf \hat{\mathtt{x}}_k^{-1}(x)\big) \leq \tilde{f}_k(z) \quad \text{for a.e. } z \in \hat{\mathtt{x}}_k^{-1}(x),$$

where $a := \int_0^{\inf \hat{\mathtt{x}}_k^{-1}(x)} \tilde{\sigma}_k(\zeta) d\zeta$. By definition of convex envelope we finally obtain

$$\tilde{\sigma}_k(z) = \sigma^* = \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(x)} \tilde{f}_k(z), \quad \text{for a.e. } z \in \hat{\mathtt{x}}_k^{-1}(x). \qquad \square$$

5.6.7. Analysis on $\tilde{v}_k$. In this section we prove that the $\tilde{v}_k$ obtained by compactness as limit of $\hat{v}_k^n$ in Proposition 5.47 satisfies the second equation in the system (5.69). As a consequence of this result and of the results obtained in the two previous sections we get that

$$\hat{u}^n \to \hat{u}, \qquad \hat{v}_k^n \to \hat{v}_k, \qquad \hat{\sigma}_k^n \to \hat{\sigma}_k,$$

for $k = 1, \dots, N$, where the convergence of $\hat{u}^n$ and $\hat{v}^n$ are in C^0 and the convergence of σ_k^n is in L^1. We need first the following remark, which adapt the results obtained in Section 5.3 to the objects we are now working on.

Remark 5.57. Let $x \in \mathbb{R}$, $\delta > 0$, $n \in \mathbb{N}$. Assume that $u^n(\bar{t}, \cdot)$ is continuous in $x - \delta$ and $x + \delta$. Denote by x_p, $p = 1, \dots, P$, the discontinuity points of u^n between $x - \delta$ and $x + \delta$. For every p, the Riemann problem $\big(u(\bar{t}, x_p-), u(\bar{t}, x_+)\big)$ is solved by a collection of N curves

$$\gamma_k^p := (u_k^p, v_k^p, \sigma_k^p), \; k = 1, \dots, N, \; p = 1, \dots, P,$$

of length s_k^p respectively. Notice that the collection of curves $\{\gamma_k^p\}_{k=1,\dots,N}^{p=1,\dots,P}$ depends on x, δ and N, even if we do not write this dependence explicitly. Observe also that, for every family k,

$$\sum_{p=1}^P s_k^p = \int_{(\mathtt{x}^n)^{-1}((x-\delta, x+\delta)) \cap (L_{k-1}, L_k]} \rho^n(w) dw \tag{5.88}$$

and

$$\sum_{p=1}^P |s_k^p| = \int_{(\mathtt{x}^n)^{-1}((x-\delta, x+\delta)) \cap (L_{k-1}, L_k]} |\rho^n(w)| dw. \tag{5.89}$$

We are exactly in the situation considered in Proposition 5.27 and Corollary 5.28. Therefore we can apply Proposition 5.27 and Corollary 5.28 and, using (5.88) and (5.89), we obtain what follows.

For every family k and for every $z \in (\hat{\mathtt{x}}_k^n)^{-1}(x - \delta, x + \delta)$, we have that

(1) the following inequality holds:

$$\begin{aligned} |\hat{v}_k^n(z)| \; \leq C \Bigg[\Bigg| \int_{(\mathtt{x}^n)^{-1}((x-\delta, x+\delta)) \cap \mathcal{W}_k} \rho^n(w) dw \Bigg| + \sum_{h \neq k} \int_{(\mathtt{x}^n)^{-1}((x-\delta, x+\delta)) \cap \mathcal{W}_h} |\rho^n(w)| dw \\ + \int_{(\hat{\mathtt{x}}_k^n)^{-1}((x-\delta, x+\delta))} \big| \hat{\sigma}_k^n(\zeta) - \sigma^* \big| d\zeta \Bigg]. \end{aligned}$$

Moreover,

(2) If

$$\int_{\mathtt{x}^{-1}(x-\delta, x+\delta) \cap W_k} \bar{\rho}^n(w) dw > 0,$$

then

$$\hat{v}_k^n(z) \geq -C\left[\sum_{h\neq k}\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h}|\rho^n(w)|dw + \int_{(\hat{\mathtt{x}}_k^n)^{-1}((x-\delta,x+\delta))}\left|\hat{\sigma}_k^n(\zeta)-\sigma^*\right|d\zeta\right].$$

(3) If

$$\int_{\mathtt{x}^{-1}(x-\delta,x+\delta)\cap W_k}\bar{\rho}^n(w)dw > 0,$$

then

$$\hat{v}_k^n(z) \leq C\left[\sum_{h\neq k}\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h}|\rho^n(w)|dw + \int_{(\hat{\mathtt{x}}_k^n)^{-1}((x-\delta,x+\delta))}\left|\hat{\sigma}_k^n(\zeta)-\sigma^*\right|d\zeta\right].$$

We can finally prove that $\tilde{v}_k$ satisfies the second equation in the system (5.69).

PROPOSITION 5.58. *For every $k=1,\dots,N$, and for every $z\in(0,M_k]$*

$$\tilde{v}_k(z) = \mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)\Big[\tilde{f}_k(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))}\tilde{f}_k(z)\Big].$$

The proof of the proposition is based on the following two lemmas.

LEMMA 5.59. *For every $k=1,\dots,N$, and for every $z\in(0,M_k]$*

$$|\tilde{v}_k(z)| = \tilde{f}_k(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))}\tilde{f}_k(z).$$

PROOF. Using Proposition 1.13, we get

$$|\hat{v}_k^n(z)| = \hat{f}_k^n(z) - \operatorname*{conv}_{(\hat{\mathtt{x}}_k^n)^{-1}(\hat{\mathtt{x}}_k^n(z)}\hat{f}_k^n(z) = \int_0^z\Big(D\hat{f}_k^n(\zeta)-\sigma_k^n(\zeta)\Big)d\zeta.$$

We can thus pass to the limit as $n\to\infty$ to get

$$|\tilde{v}_k(z)| = \int_0^z\Big(D\tilde{f}_k(\zeta)-\tilde{\sigma}_k(\zeta)\Big)d\zeta = \tilde{f}_k(\zeta) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))}\tilde{f}_k(\zeta).$$ □

LEMMA 5.60. *For every fixed family k and for every $z\notin S_k$.*

- *If $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)=+1$, then $\tilde{v}_k(z)\geq 0$.*
- *If $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)=0$, then $\tilde{v}_k(z)=0$.*
- *If $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)=-1$, then $\tilde{v}_k(z)\leq 0$.*

PROOF. Since $\tilde{v}_k$ is Lipschitz and $z\notin S_k$, we can assume w.l.o.g. that $\hat{\mathtt{x}}_k^n(z)\to\hat{\mathtt{x}}_k(z)$. Set $x:=\hat{\mathtt{x}}_k(z)$ and fix $\delta>0$, $n\in\mathbb{N}$. Suppose that

$$\mathcal{L}^1\Big(\mathtt{x}^{-1}(x-\delta)\Big) = \mathcal{L}^1\Big(\mathtt{x}^{-1}(x+\delta)\Big) = 0 \tag{5.90}$$

and

$$u^n \text{ is continuous at } x-\delta, x+\delta \text{ for every } n\in\mathbb{N}. \tag{5.91}$$

Assume now first that $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)=+1$. We want to prove that $\tilde{v}_k(z)\geq 0$. Since $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)=+1$, by definition

$$\int_{\mathtt{x}^{-1}(x)\cap(L_{k-1},L_k]}\rho(w)dw > 0.$$

and thus, if $\delta\ll 1$ and $n\gg 1$ (depending on δ),

$$\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap(L_{k-1},L_k]}\rho^n(w)dw > 0$$

and $\hat{\mathtt{x}}^n_k(z) \in (x-\delta, x+\delta)$. Therefore, by Remark 5.57, Point (2), there exists a constant $C>0$, depending only on f, such that for every constant $\sigma^* \in \mathbb{R}$,

$$\hat{v}^n_k(z) \geq -C\left[\sum_{h\neq k}\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} |\rho^n(w)|dw + \int_{(\hat{\mathtt{x}}^n_k)^{-1}((x-\delta,x+\delta))} \big|\hat{\sigma}^n_k(\zeta)-\sigma^*\big|d\zeta\right].$$

For fixed $\delta>0$, the previous relation holds for every $n \gg 1$. We can thus pass to the limit as $n\to\infty$. Using (5.90) and Lemma 1.21, we get

$$\tilde{v}_k(z) \geq -C\left[\sum_{h\neq k}\int_{\mathtt{x}^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} \bar{\rho}(w)dw + \int_{\hat{\mathtt{x}}^{-1}_k((x-\delta,x+\delta))} \big|\hat{\sigma}_k(\zeta)-\sigma^*\big|d\zeta\right]. \tag{5.92}$$

We have just proved that if (5.90) and (5.91) hold, then (5.92) holds too. Since (5.90) and (5.91) hold for a.e. $\delta>0$, we can pass to the limit as $\delta\to 0$ in (5.92) to get

$$\tilde{v}_k(z) \geq -C\left[\sum_{h\neq k}\int_{\mathtt{x}^{-1}(x)\cap\mathcal{W}_h} \bar{\rho}(w)dw + \int_{\hat{\mathtt{x}}^{-1}_k(x)} \big|\hat{\sigma}_k(\zeta)-\sigma^*\big|d\zeta\right].$$

By Lemma 5.41 the first term in the r.h.s. is zero and by Corollary 5.55 also the second term in the r.h.s. is zero for a suitable choice of σ^* and thus $v_k(z)\geq 0$. The case $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big) = -1$ is completely similar.

We prove now that if $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)=0$ then $\tilde{v}_k(z)=0$. Using Remark 5.57, Point (1), we have that there exists a constant $C>0$ depending only on f such that for every $\delta>0$ satisfying (5.90) and (5.91) and for every $n\gg 1$ (depending on δ),

$$\begin{aligned}|\hat{v}^n_k(z)| \ &\leq C\Bigg[\bigg|\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_k} \rho^n(w)dw\bigg| + \sum_{h\neq k}\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} |\rho^n(w)|dw \\ &\qquad + \int_{(\hat{\mathtt{x}}^n_k)^{-1}((x-\delta,x+\delta))} \big|\hat{\sigma}^n_k(\zeta)-\sigma^*\big|d\zeta\Bigg].\end{aligned}$$

As before, we can pass to the limit as $n\to\infty$ to get

$$\begin{aligned}|\tilde{v}_k(z)| \ &\leq C\Bigg[\bigg|\int_{\mathtt{x}^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_k} \rho(w)dw\bigg| + \sum_{h\neq k}\int_{\mathtt{x}^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} |\rho(w)|dw \\ &\qquad + \int_{\hat{\mathtt{x}}^{-1}_k((x-\delta,x+\delta))} \big|\hat{\sigma}_k(\zeta)-\sigma^*\big|d\zeta\Bigg].\end{aligned}$$

and as $\delta\to 0$,

$$|\tilde{v}_k(z)| \ \leq C\Bigg[\bigg|\int_{\mathtt{x}^{-1}(x)\cap\mathcal{W}_k} \rho(w)dw\bigg| + \sum_{h\neq k}\int_{\mathtt{x}^{-1}(x)\cap\mathcal{W}_h} |\rho(w)|dw + \int_{\hat{\mathtt{x}}^{-1}_k(x)} \big|\hat{\sigma}_k(\zeta)-\sigma^*\big|d\zeta\Bigg].$$

The first term in the r.h.s. is exactly $\mathcal{S}_k(x)$ and thus it is zero, and, as before, also the second and the third term are zero. Therefore, if $\mathcal{S}_k(x)=0$, then $\tilde{v}_k(z)=0$. ☐

Proof of Proposition 5.58. If $z\in S_k$, then

$$\tilde{f}_k(z) - \operatorname*{conv}_{\hat{\mathtt{x}}^{-1}_k(\hat{\mathtt{x}}_k(z))} \tilde{f}_k(z) = 0$$

and thus by Lemma 5.59

$$\tilde{v}_k(z) = |\tilde{v}_k(z)| = \tilde{f}_k(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} \tilde{f}_k(z) = 0.$$

Assume thus that $z \notin S_k$.

- If $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big) = +1$, then by Lemma 5.60, $\tilde{v}_k(z) \geq 0$. Therefore, using Lemma 5.59,

$$\tilde{v}_k(z) = |\tilde{v}_k(z)| = \tilde{f}_k(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} \tilde{f}_k(z) = \mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)\Big[\tilde{f}_k(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} \tilde{f}_k(z)\Big].$$

- If $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big) = 0$, then by Lemma 5.60

$$\tilde{v}_k(z) = 0 = \mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big)\Big[\tilde{f}_k(z) - \operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(\hat{\mathtt{x}}_k(z))} \tilde{f}_k(z)\Big].$$

- If $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big) = -1$, then one argues as in the case $\mathcal{S}_k\big(\hat{\mathtt{x}}_k(z)\big) = +1$. □

5.7. Proof of Properties (b), (c), (e) and (5.29)

In Section 5.5 we proved the existence of a $(N+4)$-tuple

$$\mathcal{E} := (L_0, \dots, L_N, \mathtt{x}, \rho, \bar{\rho})$$

where

$$L_0 \leq \dots \leq L_N, \qquad \mathtt{x} : [0,T] \times (L_0, L_N] \to \mathbb{R}, \qquad \rho, \bar{\rho} : [0,T] \times (L_0, L_N] \to [-1,1],$$

such that

a) for every time $t \in [0,T]$, the collection

$$\mathcal{E}(t) := (L_0, \dots, L_N, \mathtt{x}(t,\cdot), \rho(t,\cdot), \bar{\rho}(t,\cdot))$$

is an e.o.w. (Property (a) in the definition of Lagrangian representation, Definition 5.21);

b) the distributions $D_t\rho$, $D_t\bar{\rho}$ are finite Radon measure on $[0,T] \times \mathbb{R}$ (property (d) in Definition 5.21).

In order to complete the proof of Theorem C, we need still to show that Properties (b), (c), (e) in Definition 5.21 and the additional property (5.29) holds. This is the aim of this section.

5.7.1. Proof of Property (b). We start proving Property (b) together with two corollaries of its, which will be used later.

THEOREM 5.61. *For any time $\bar{t} \notin \mathcal{Z}$, it holds*

$$D_x u(\bar{t}) = \mathtt{x}_\sharp\big(\rho(\bar{t}) r^{\hat{\gamma}(\bar{t})} \mathcal{L}^1\big).$$

PROOF. By Corollary 5.30, we already know that, for the approximations,

$$D_x u^n(\bar{t}) = \hat{\mathtt{x}}(\bar{t})_\sharp\big(\rho^n(\bar{t}) r^{\hat{\gamma}^n(\bar{t})} \mathcal{L}^1|_{(L_0, L_N]}\big).$$

The conclusion can now be easily obtained, passing to the limit as $n \to \infty$ as arguing as in the proof of Proposition 5.51. □

COROLLARY 5.62. *For every $z \in [0,M]$, setting $x := \hat{\mathtt{x}}(z)$, we have*

$$u(\bar{t}, x-) = \hat{u}\big(\bar{t}, \inf \hat{\mathtt{x}}^{-1}(x)\big), \qquad u(\bar{t}, x+) = \hat{u}\big(\bar{t}, \sup \hat{\mathtt{x}}^{-1}(x)\big).$$

PROOF. It holds

$$u(\bar t, x-) = D_x u\big((-\infty, x)\big), \qquad u(\bar t, x+) = D_x u\big((-\infty, x]\big).$$

Therefore

$$u(\bar t, x-) = D_x u\big((-\infty, x)\big) = \int_{\hat{\mathtt{x}}^{-1}((-\infty,x))} D_z \hat u(\zeta) d\zeta = \int_0^{\inf \hat{\mathtt{x}}^{-1}(x)} D_z u(\zeta) d\zeta = \hat u\big(\bar t, \inf \hat{\mathtt{x}}^{-1}(x)\big).$$

Similarly

$$u(\bar t, x+) = \hat u\big(\bar t, \sup \hat{\mathtt{x}}^{-1}(x)\big). \qquad \square$$

REMARK 5.63. The previous corollary also implies that

- if x is a continuity point of $u(\bar t, \cdot)$, then either $\hat{\mathtt{x}}^{-1}(x)$ contains a single point, or it is an interval and $\hat u|_{\hat{\mathtt{x}}^{-1}(x)}$ is a closed curve starting and ending at $u(\bar t, x)$;
- if x is a jump point of $u(\bar t, \cdot)$, then $\hat{\mathtt{x}}^{-1}(x)$ is an interval and $\hat u|_{\hat{\mathtt{x}}^{-1}(x)}$ is a curve starting at $u(\bar t, x-)$ and ending at $u(\bar t, x+)$.

COROLLARY 5.64. *For every $x \in \mathbb{R}$,*

$$x \text{ is a continuity point for } u(\bar t, \cdot) \iff \int_{\mathtt{x}(\bar t)^{-1}(x)} \rho(\bar t, w) r^{\hat\gamma(\bar t)}(w) dw = 0.$$

PROOF. It easily follows from the previous corollary, (5.17) and the fact that

$$V(\bar t)^{-1}\big([\inf \hat{\mathtt{x}}(\bar t)^{-1}(x), \sup \hat{\mathtt{x}}((\bar t)^{-1}(x)]\big) \triangle \mathtt{x}(\bar t)^{-1}(x)$$

is $\bar\rho(\bar t)\mathcal{L}^1$-negligible. $\square$

5.7.2. Proof of Property (c) and proof of (5.29). We continue in this section the proof of Theorem C, proving that our candidate Lagrangian representation

$$\mathcal{E} := (L_0, \dots, L_N, \mathtt{x}, \rho, \bar\rho)$$

satisfies Property (c) in Definition 5.21 and also the additional property (5.29). We start our analysis with the following remark which studies the behavior of a Glimm approximate solution u^n at a fixed time $\bar t$ in an interval $[x-\delta, x+\delta]$. We again omit to write the explicit time dependence, since we work at fixed time $\bar t \notin \mathcal{Z}$.

REMARK 5.65. Let $x \in \mathbb{R}$, $\delta > 0$, $n \in \mathbb{N}$. Assume that $u^n(\bar t, \cdot)$ is continuous in $x - \delta$ and $x + \delta$. As in Remark 5.57, denote by x_p, $p = 1, \dots, P$, the discontinuity points of u^n between $x - \delta$ and $x + \delta$. For every p, the Riemann problem $\big(u(\bar t, x_p-), u(\bar t, x_+)\big)$ is solved by a collection of N curves

$$\gamma_k^p := (u_k^p, v_k^p, \sigma_k^p), \ k = 1, \dots, N, \ p = 1, \dots, P.$$

of length s_k^p respectively. Notice that the collection of curves $\{\gamma_k^p\}_{k=1,\dots,N}^{p=1,\dots,P}$ depends on x, δ and N, even if we do not write this dependence explicitly. Observe also that, for every family k,

$$\sum_{p=1}^{P} s_k^p = \int_{(\mathtt{x}^n)^{-1}((x-\delta, x+\delta)) \cap (L_{k-1}, L_k]} \rho^n(w) dw \tag{5.93}$$

and

$$\sum_{p=1}^{P} |s_k^p| = \int_{(\mathtt{x}^n)^{-1}((x-\delta, x+\delta)) \cap (L_{k-1}, L_k]} |\rho^n(w)| dw. \tag{5.94}$$

We are exactly in the situation considered in Proposition 5.27 and Corollary 5.28. Therefore we can apply Proposition 5.27 and Corollary 5.28 together with (5.93) and (5.94), to construct, for every family k, a curve

$$\gamma_k^{n,\delta} := \mathcal{G}_k\Big(\{\gamma_k^p\}_k^p\Big) = \big(u_k^{n,\delta}, v_k^{n,\delta}, \sigma_k^{n,\delta}\big)$$

(see (5.51)) of length

$$s_k^{n,\delta} := \sum_p s_k^p$$

connecting

$$\phi_k^{n,\delta} := u_k^{n,\delta}(0), \qquad \psi_k^{n,\delta} := u_k^{n,\delta}(s_k^{n,\delta}),$$

with the following properties. For every constant $\sigma^* \in \mathbb{R}$

$$\big|u^n(\bar t, x-\delta) - \phi_k^{n,\delta}\big| \leq \mathcal{O}(1)\sum_{h\neq k}\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} |\rho^n(w)|dw, \tag{5.95a}$$

$$\begin{aligned}&\big|u^n(\bar t, x+\delta) - \psi_k^{n,\delta}\big| \\ &\quad\leq \mathcal{O}(1)\sum_{h\neq k}\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} |\rho^n(w)|dw + \int_{(\hat{\mathtt{x}}_k^n)^{-1}((x-\delta,x+\delta))} \big|\hat\sigma_k^n(\zeta) - \sigma^*\big|d\zeta,\end{aligned} \tag{5.95b}$$

$$\begin{aligned}&\big\|\sigma_k^{n,\delta} - \sigma^*\big\|_{L^1(\mathbf{I}(s_k^{n,\delta}))} \\ &\quad\leq \mathcal{O}(1)\sum_{h\neq k}\int_{(\mathtt{x}^n)^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} |\rho^n(w)|dw + \int_{(\hat{\mathtt{x}}_k^n)^{-1}((x-\delta,x+\delta))} \big|\hat\sigma_k^n(\zeta) - \sigma^*\big|d\zeta.\end{aligned} \tag{5.95c}$$

We need now the following two lemmas.

Lemma 5.66. *Let $x \in \mathbb{R}$ be a fixed point. Then*

(a) the (possibly trivial) Riemann problem $\big(u(\bar t, x-), u(\bar t, x+)\big)$ is solved by the curves $\{\gamma_k\}$, $k = 1, \dots, N$, where $\gamma_k = (u_k, v_k, \sigma_k)$ in an exact curve of the k-th family of lenght

$$s_k := \int_{\hat{\mathtt{x}}^{-1}(x)\cap\mathcal{W}_k} \rho(w)dw;$$

(b) for every $\tau \in \boldsymbol{I}(s_k)$ and $z \in \hat{\mathtt{x}}_k^{-1}(z)$ it holds $\sigma_k(\tau) = \hat\sigma_k(z)$.

Remark 5.67. If $s_k = 0$ for any k, the proposition means that $u(\bar t, \cdot)$ is continuous at x. If $s_k \neq 0$ for some k, the proposition, together with Corollary 5.42 means that $u(\bar t, \cdot)$ has a jump in x and the Riemann problem located at $(\bar t, x)$ is solved by a single discontinuity (made by shocks or contact discontinuities of the k-th family) moving with speed σ^*.

Proof. Fix $\delta > 0$, $n \in \mathbb{N}$. Suppose that

$$\mathcal{L}^1\Big(\mathtt{x}^{-1}(x-\delta)\Big) = \mathcal{L}^1\Big(\mathtt{x}^{-1}(x+\delta)\Big) = 0, \tag{5.96a}$$

$$u^n \text{ is continuous at } x-\delta, x+\delta \text{ for every } n \in \mathbb{N} \tag{5.96b}$$

and

$$u^n(\bar t, x-\delta) \to u(\bar t, x-\delta), \qquad u^n(\bar t, x+\delta) \to u(\bar t, x+\delta). \tag{5.96c}$$

The conditions in (5.96) are satisfied for a.e. $\delta > 0$. We already know that there exists a curve $\gamma_k^{n,\delta}$ with the properties described in Remark 5.65. Using the same notations as in Remark 5.65, we have that, by compactness, as $n \to \infty$, $\phi_k^{n,\delta}$ converges to some ϕ_k^δ and $\psi_k^{n,\delta}$

converges to some ψ_k^δ, up to subsequences (which we do not explicitly denote). Notice also that $s_k^{n,\delta} \to s_k^\delta$, where

$$s_k^\delta = \int_{\mathtt{x}^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_k} \rho(w)dw.$$

Thus, by Lemma 3.15, the collection of curves $\{\gamma_k^{n,\delta}\}$ converges, as $n \to \infty$, to a curve $\gamma_k^\delta = (u_k^\delta, v_k^\delta, \sigma_k^\delta)$ of length s_k^δ, which connects the left state ϕ_k^δ with the right state ψ_k^δ. Passing to the limit in (5.95), we get the following estimates:

$$\big|u(\bar t, x-\delta) - \phi_k^\delta\big| \le \mathcal{O}(1) \sum_{h\neq k} \int_{\mathtt{x}^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} \bar\rho(w)dw, \tag{5.97a}$$

$$\big|u(\bar t, x+\delta) - \psi_k^\delta\big| \le \mathcal{O}(1) \Bigg[\sum_{h\neq k} \int_{\mathtt{x}^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} \bar\rho(w)dw + \int_{\hat{\mathtt{x}}_k^{-1}(x-\delta,x+\delta)} \big|\hat\sigma_k(z) - \sigma^*\big|dz \Bigg], \tag{5.97b}$$

$$\big\|\sigma_k^\delta - \sigma^*\big\|_{L^1(\mathbf{I}(s_k^\delta))} \le \mathcal{O}(1) \Bigg[\sum_{h\neq k} \int_{\mathtt{x}^{-1}((x-\delta,x+\delta))\cap\mathcal{W}_h} \bar\rho(w)dw + \int_{\hat{\mathtt{x}}_k^{-1}(x-\delta,x+\delta)} \big|\hat\sigma_k(z) - \sigma^*\big|dz \Bigg], \tag{5.97c}$$

for every constant $\sigma^* \in \mathbb{R}$. Therefore, as $\delta \to 0$:

- the r.h.s. of (5.97a) tends to zero, because of Corollary 5.42;
- choosing σ^* equal to the speed given by Corollary 5.55 and using again Corollary 5.42, the r.h.s. of (5.97b) tends to zero;
- $s_k^\delta \to s_k$.

Hence, the collection of curves $\{\tilde\gamma_k^\delta\}$ converges to an exact curve $\gamma_k = (u_k, v_k, \sigma_k)$ of length s_k which connects $u(\bar t, x-)$ to $u(\bar t, x+)$. Moreover, if $s_k \neq 0$, passing to the limit (5.97c), we get

$$\big\|\sigma_k - \sigma^*\big\|_{L^1(\mathbf{I}(s_k))} \le \mathcal{O}(1) \Bigg[\sum_{h\neq k} \int_{\mathtt{x}^{-1}(x)\cap\mathcal{W}_h} \bar\rho(w)dw + \int_{\hat{\mathtt{x}}_k^{-1}(x)} \big|\hat\sigma_k(z) - \sigma^*\big|dz \Bigg].$$

The proof now is completed observing that, by Corollary 5.42, for every $h \neq k$,

$$\int_{\mathtt{x}^{-1}(x)\cap\mathcal{W}_h} \bar\rho(w)dw = \mathcal{L}^1\Big(\hat{\mathtt{x}}_k^{-1}(x)\Big) = 0,$$

while, by Corollary 5.55,

$$\int_{\hat{\mathtt{x}}_k^{-1}(x)} \big|\hat\sigma_k(z) - \sigma^*\big|dz = 0. \qquad \square$$

LEMMA 5.68. *For a.e. $z \in S_k$, setting $x := \hat{\mathtt{x}}_k(z)$,*

$$\hat u\big(\omega_k(z)\big) = u(\bar t, x).$$

PROOF. By Lemma 5.16, for a.e. $z \in (0, M_k]$, if $z \in S_k$, then $\omega_k(z) \in S$ and thus

$$\int_{\mathtt{x}^{-1}(x)} \bar\rho(w)dw = 0.$$

Therefore, by Corollary 5.64, x is a continuity point for $u(\bar t,\cdot)$. We thus have

$$\begin{aligned}
\hat u\big(\omega_k(z)\big) &= \int_0^{\omega_k(z)} D_z\hat u(\zeta)d\zeta \\
&= \int_0^{\omega_k(z)} V_\sharp\big(\rho r^{\hat\gamma}\mathcal{L}^1\big)(d\zeta) \\
&= \int_{V^{-1}((0,\omega_k(z)])} \rho(w)r^{\hat\gamma}(w)dw \\
\text{(by Lemma 5.17)} &= \int_{\mathtt{x}^{-1}((-\infty,x])} \rho(w)r^{\hat\gamma}(w)dw \\
&= \int_{(-\infty,x]} \mathtt{x}_\sharp\big(\rho r^{\hat\gamma}\mathcal{L}^1\big)(dx) \\
\text{(by Theorem 5.61)} &= u(\bar t,x).
\end{aligned}$$

□

We can finally state and prove the following theorem, which, together with its corollary, proves Property (c) in the Definition of Lagrangian representation, Definition 5.21.

THEOREM 5.69. *For every $k=1,\dots,N$ and for a.e. $z\in(0,M_k]$, it holds*

$$\hat\sigma_k(\bar t,z)=\lambda_k\big(\bar t,\hat{\mathtt{x}}_k(z)\big). \tag{5.98}$$

PROOF. As before, since we are working at fixed time $\bar t\notin\mathcal{Z}$, we will omit to write explicitly the time dependence. We separately consider the following three cases.

(1) For a.e. $z\in S_k$, (5.98) holds.
(2) For every $x\in\mathbb{R}$ such that

$$\operatorname{card}\hat{\mathtt{x}}_k^{-1}(x)>1 \text{ and } \int_{\mathtt{x}^{-1}(x)}\rho(w)dw=0 \tag{5.99}$$

(5.98) holds for a.e. $z\in\hat{\mathtt{x}}_k^{-1}(x)$.
(3) For every $x\in\mathbb{R}$ such that

$$\operatorname{card}\hat{\mathtt{x}}_k^{-1}(x)>1 \text{ and } \int_{\mathtt{x}^{-1}(x)}\rho(w)dw\neq 0$$

(5.98) holds for a.e. $z\in\hat{\mathtt{x}}_k^{-1}(x)$.

Let us thus start with the proof of the first point. For a.e. $z\in S_k$

$$\begin{aligned}
\hat\sigma_k(z) &= D\hat f_k(z) \\
&= \tilde\lambda_k\big(\hat u(\omega_k(z)),\hat v_k(z),\hat\sigma_k(z)\big) \\
\text{(since } z\in S_k \text{ and thus } \hat v_k(z)=0) &= \lambda_k\big(\hat u(\omega_k(z))\big) \\
\text{(by Lemma 5.68)} &= \lambda_k\big(u(\bar t,\mathtt{x}_k(z)\big).
\end{aligned}$$

Let us prove now the second point. Take any $x\in\mathbb{R}$ such that (5.99) holds. By Lemma 5.60, for every $z\in\hat{\mathtt{x}}_k^{-1}(x)$ $\hat v_k(z)=0$ and thus $\hat f_k(z)=\operatorname{conv}_{\hat{\mathtt{x}}_k^{-1}(x)}\hat f_k(z)$. Therefore, by Corollary 5.55, for a.e. $z\in\hat{\mathtt{x}}_k^{-1}(x)$,

$$D\hat f_k(z)=D\operatorname*{conv}_{\hat{\mathtt{x}}_k^{-1}(x)}\hat f_k(z)=\hat\sigma_k(z)=\sigma^*.$$

On the other side, for a.e. $z \in \hat{\mathtt{x}}_k^{-1}(x)$,

$$D\hat{f}_k(z) = \tilde{\lambda}_k\Big(\hat{u}\big(\omega_k(z)\big), \hat{v}_k(z), \hat{\sigma}_k(z)\Big) = \lambda_k\Big(\hat{u}\big(\omega_k(z)\big)\Big).$$

Therefore, for a.e. $z \in \hat{\mathtt{x}}_k^{-1}(x)$,

$$\sigma^* = \hat{\sigma}_k(z) = \lambda_k\Big(\hat{u}\big(\omega_k(z)\big)\Big). \tag{5.100}$$

Notice that by Proposition 5.71, $u(\bar{t}, \cdot)$ is continuous at x. Therefore, by Remark 5.63, Corollary 5.55 and Lemma 5.14, $\hat{u} \circ \omega_k$ is a continuous closed curve starting and ending at $u(\bar{t}, x)$. We can thus pass to the limit in (5.100) as $z \to \inf \hat{\mathtt{x}}_k^{-1}(x)$ (or $z \to \sup \hat{\mathtt{x}}_k^{-1}(x)$) to get for a.e. $z \in \hat{\mathtt{x}}_k^{-1}(x)$,

$$\sigma^* = \hat{\sigma}_k(z) = \lambda_k\Big(u(\bar{t}, x)\Big).$$

The third case is an immediate consequence of Lemma 5.66. □

Corollary 5.70. *For $\bar{\rho}(\bar{t})\mathcal{L}^1$-a.e. $w \in \mathcal{W}_k$,*

$$\hat{\sigma}_k\big(\bar{t}, V_k(\bar{t}, w)\big) = \lambda_k(\bar{t}, \mathtt{x}(\bar{t}, w)).$$

Proof. We have

$$\begin{aligned}
0 &= \int_0^{M_k(\bar{t})} \Big|\hat{\sigma}_k(\bar{t}, z) - \lambda_k\big(\bar{t}, \hat{\mathtt{x}}_k(\bar{t}, z)\big)\Big| dz \\
\text{(changing variable: } z = V_k(\bar{t}, w)\text{)} &= \int_{L_{k-1}}^{L_k} \Big|\hat{\sigma}_k\big(\bar{t}, V_k(\bar{t}, w)\big) - \lambda_k\Big(\bar{t}, \hat{\mathtt{x}}_k\big(\bar{t}, \big(V_k(\bar{t}, w)\big)\big)\Big)\Big| \bar{\rho}(\bar{t}, w) dw \\
&= \int_{L_{k-1}}^{L_k} \Big|\hat{\sigma}_k\big(\bar{t}, V_k(\bar{t}, w)\big) - \lambda_k\Big(\bar{t}, \hat{\mathtt{x}}_k(\bar{t}) \circ V_k(\bar{t})(w)\Big)\Big| \bar{\rho}(\bar{t}, w) dw \\
\text{(by Proposition 5.11)} &= \int_{L_{k-1}}^{L_k} \Big|\hat{\sigma}_k\big(\bar{t}, V_k(\bar{t}, w)\big) - \lambda_k\Big(\bar{t}, \mathtt{x}(\bar{t}, w)\Big)\Big| \bar{\rho}(\bar{t}, w) dw.
\end{aligned}$$

□

The last theorem of this section concludes the proof of (5.29) in the statement of Theorem C. Its proof is an immediate consequence of Lemma 5.66.

Theorem 5.71. *For every $x \in \mathbb{R}$,*

$$x \textit{ is a continuity point for } u(\bar{t}, \cdot) \iff \int_{\mathtt{x}(\bar{t})^{-1}(x)} \rho(w) dw = 0.$$

5.7.3. Proof of Property (e): the characteristic equation. We finally prove that also Property (e) in the definition of Lagrangian representation, Definition 5.21 is satisfied. This concludes the proof of Theorem C.

Theorem 5.72. *For a.e. fixed wave $w \in (L_0, L_N]$ the map $t \mapsto \mathtt{x}(t, w)$ is 1-Lipschitz and therefore is it differentiable for a.e. time $t \in [0, T]$; moreover*

$$\frac{\partial \mathtt{x}}{\partial t}(t, w) = \lambda_k(t, \mathtt{x}(t, w)), \ \textit{for } \bar{\rho}(w)\mathcal{L}^1\textit{-a.e. time } t \in [0, T].$$

Proof. We already know from Proposition 5.38 that for a.e. wave $w \in (L_0, L_N]$, the map $t \mapsto \mathtt{x}(t, w)$ is Lipschitz. Therefore for a.e. time $t \in [0, T]$ and for a.e. wave $w \in (L_0, L_N]$, the derivative $\frac{\partial \mathtt{x}}{\partial t}(t,w)$ exists. Take thus any time $\bar{t} \notin Z$ such that for a.e. wave $w \in (L_0, L_N]$, the derivative $\frac{\partial \mathtt{x}}{\partial t}(t,w)$ exists. Take any other time $t > \bar{t}$. By Lemma 5.53, it holds

$$\int_{(L_{k-1},L_k]\cap \mathtt{x}(\bar{t})^{-1}(x)} \left| \frac{\mathtt{x}(t,w) - \mathtt{x}(\bar{t},w)}{t-\bar{t}} - \sigma(\bar{t},w) \right| \bar{\rho}(\bar{t},w)dw \leq 4C\mu\Big(\big[\min\{\bar{t},t\},\max\{\bar{t},t\}\big] \times \mathbb{R}\Big),$$

where we set, for simplicity,

$$\sigma(\bar{t},w) := \sigma_k\big(\bar{t}, V_k(\bar{t},w)\big)$$

for $w \in \mathcal{W}_k$. Therefore, taking the limit as $t \to \bar{t}$ and using the fact that $\bar{t} \notin \mathcal{Z}$, we get

$$\int_{(L_{k-1},L_k]\cap \mathtt{x}(\bar{t})^{-1}(x)} \left| \frac{\partial x}{\partial t}(\bar{t},w) - \sigma(\bar{t},w) \right| \bar{\rho}(\bar{t},w)dw = 0.$$

By Corollary 5.70,

$$\int_{(L_{k-1},L_k]\cap \mathtt{x}(\bar{t})^{-1}(x)} \left| \frac{\partial x}{\partial t}(\bar{t},w) - \lambda_k(t,\mathtt{x}(t,w)) \right| \bar{\rho}(\bar{t},w)dw = 0. \tag{5.101}$$

Since (5.101) holds for a.e. time, we can integrate over time and switch the integrals to get

$$\int_{(L_{k-1},L_k]\cap \mathtt{x}(\bar{t})^{-1}(x)} \int_0^T \left| \frac{\partial x}{\partial t}(\bar{t},w) - \lambda_k(t,\mathtt{x}(t,w)) \right| \bar{\rho}(\bar{t},w)dtdw = 0.$$

Hence for a.e. fixed wave $w \in (L_{k-1}, L_k]$,

$$\frac{\partial \mathtt{x}}{\partial t}(t,w) = \lambda_k(t,\mathtt{x}(t,w)), \text{ for } \bar{\rho}(w)\mathcal{L}^1\text{-a.e. time } t \in [0,T],$$

which is what we wanted to obtain. $\square$

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